

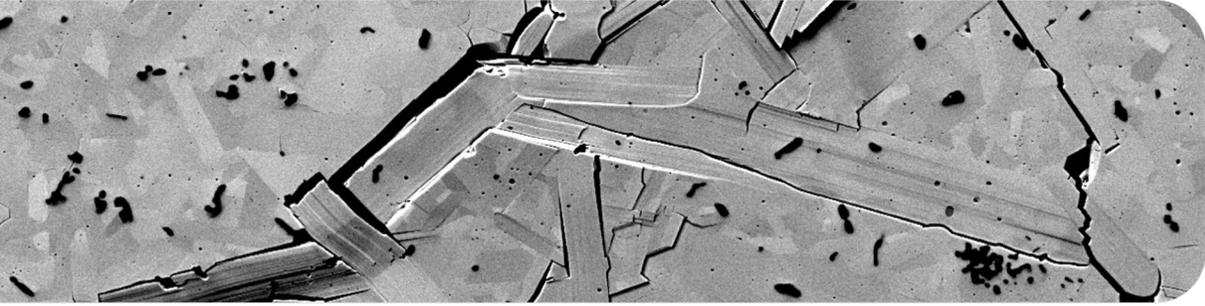


# COE–3001: Mechanics of deformable bodies

## Chapter 5: bending

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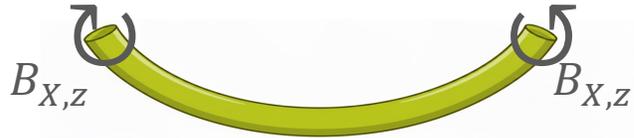


## Deformation measurement

# Deformation measure: curvature

❖ **Reminder:**

$$\{J_X^{coh,bending}\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_{X,z} \end{Bmatrix}$$



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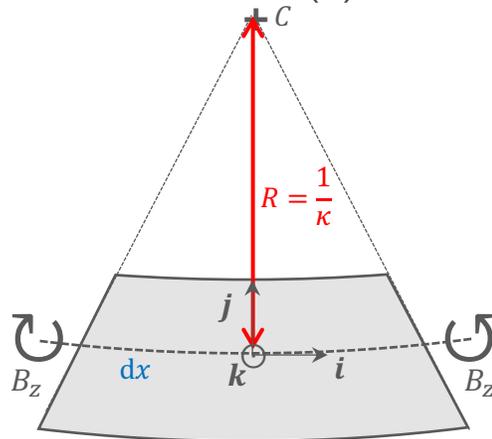
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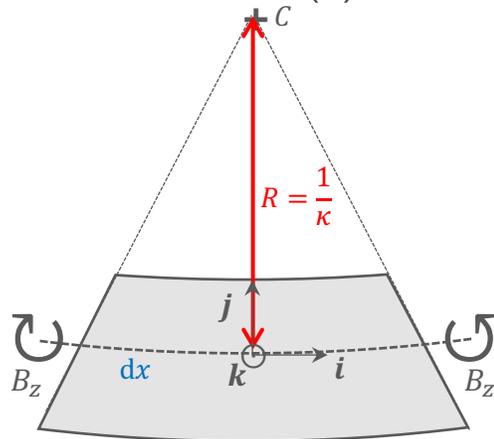
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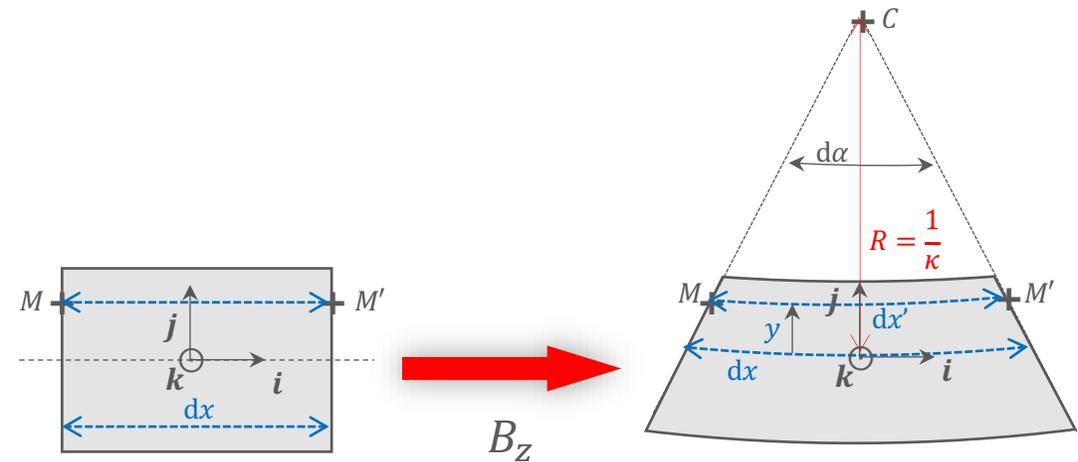
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## ❖ Infinitesimal rotation of the cross-section:

$$d\alpha = \frac{dx}{R} \Leftrightarrow \frac{d\alpha}{dx} = \kappa(x)$$

$$dx' = (R - y)d\alpha$$



# Deformation measure: curvature

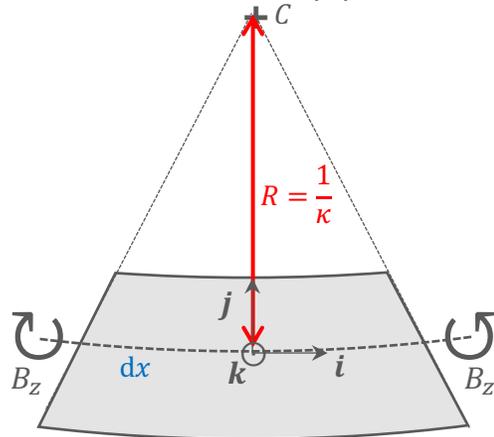
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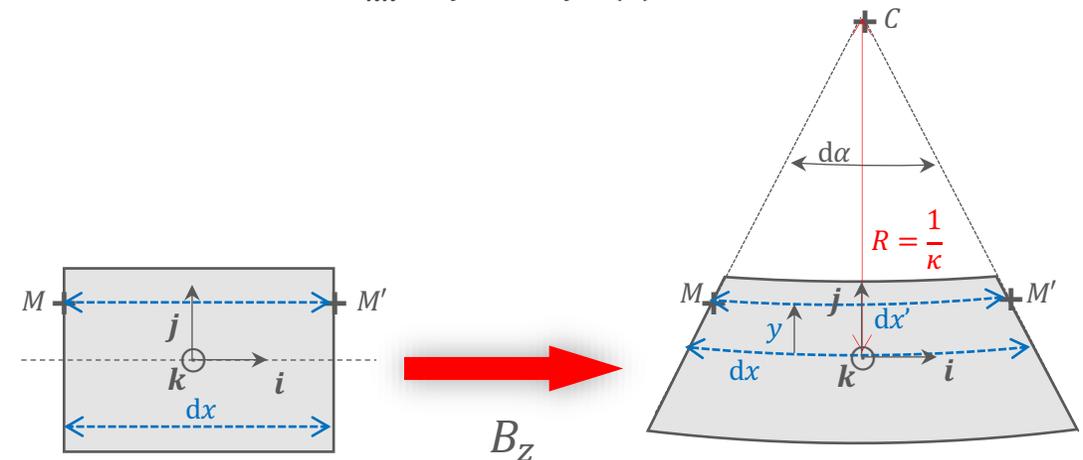
## ❖ Axial strain:

- The axial strain ( $\varepsilon_{xx}$ ) varies along  $i$  and across  $j$ .

$$\varepsilon_{xx}(x, y) = \frac{dx' - dx}{dx} = \frac{(R - y)d\alpha - R d\alpha}{R d\alpha}$$

$$\Rightarrow \varepsilon_{xx}(x, y) = -\frac{y}{R}$$

$$\Rightarrow \varepsilon_{xx}(x, y) = -y\kappa(x)$$



# Deformation measure: curvature

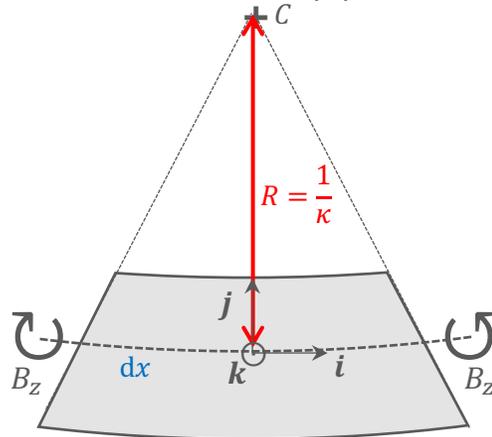
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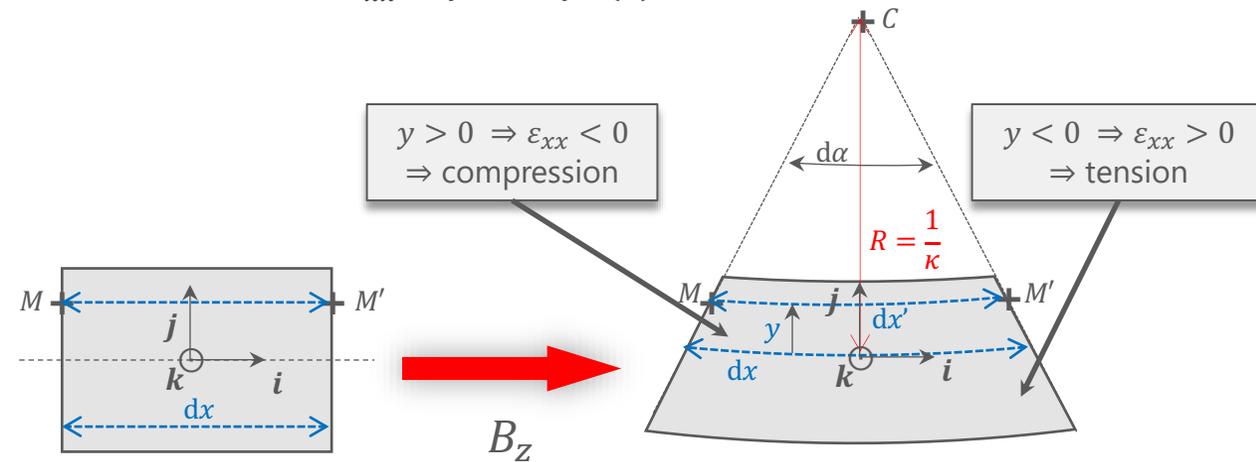
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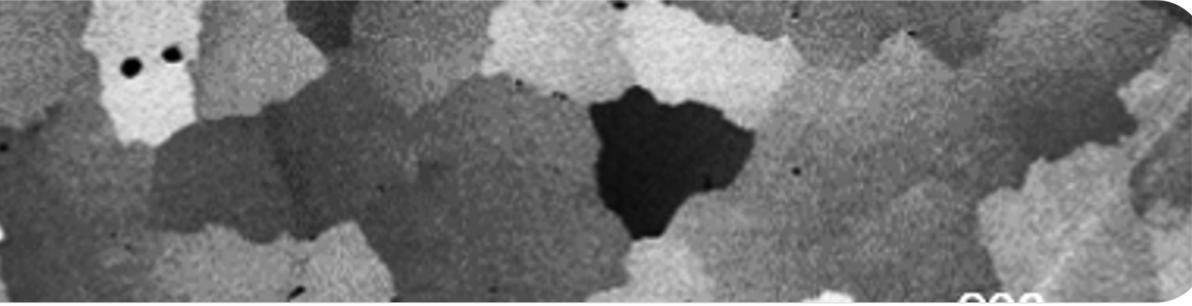
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# Stress

# Relationship between $\phi$ and the internal forces

- ❖ Cohesion torsor in  $M$ :

$$\{\mathcal{T}_M^{coh}\} = \begin{cases} \mathbf{R}_M = \mathbf{dF} \\ \mathbf{M}_M = \mathbf{0} \end{cases}$$

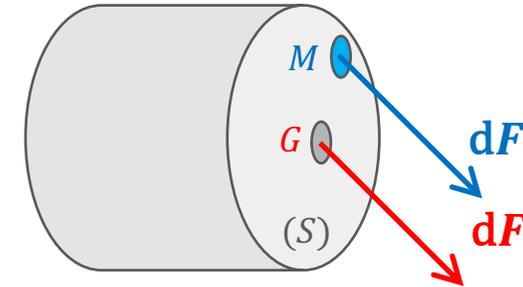
- ❖ Let's transport this torsor into  $G$  near  $M$ :

$$\{\mathcal{T}_G^{coh}\} = \begin{cases} \mathbf{dR}_G = \mathbf{R}_M = \mathbf{dF} \\ \mathbf{dM}_G = \mathbf{0} + \mathbf{GM} \times \mathbf{R}_M \end{cases} = \begin{cases} \mathbf{dR}_G = \mathbf{dF} \\ \mathbf{dM}_G = \mathbf{GM} \times \mathbf{dF} \end{cases}$$

$$\text{BUT } \phi(M, \mathbf{n}) = \frac{\mathbf{dF}}{dS} \Rightarrow \mathbf{dF} = \phi(M, \mathbf{n})dS$$

$$\Rightarrow \{\mathcal{T}_G^{coh}\} = \begin{cases} \mathbf{dR}_G = \phi(M, \mathbf{n})dS \\ \mathbf{dM}_G = \mathbf{GM} \times \phi(M, \mathbf{n})dS \end{cases}$$

$$\text{BUT } \phi(M, \mathbf{n} = \mathbf{i}) = \sigma_{xx}\mathbf{i} + \sigma_{xy}\mathbf{j} + \sigma_{xz}\mathbf{k} \text{ AND } \mathbf{GM} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



- ❖ After integration over  $(S)$ :

$$\{\mathcal{T}_G^{coh}\} = \begin{cases} \mathbf{R}_G = \iint (\sigma_{xx}\mathbf{i} + \sigma_{xy}\mathbf{j} + \sigma_{xz}\mathbf{k})dS \\ \mathbf{M}_G = \iint \mathbf{GM} \times (\sigma_{xx}\mathbf{i} + \sigma_{xy}\mathbf{j} + \sigma_{xz}\mathbf{k})dS \end{cases} = \begin{cases} N_G = \iint \sigma_{xx}dS & T_G = \iint (y\sigma_{xz} - z\sigma_{xy})dS \\ S_{G,y} = \iint \sigma_{xy}dS & B_{G,y} = \iint (z\sigma_{xz} - x\sigma_{xz})dS \\ S_{G,z} = \iint \sigma_{xz}dS & B_{G,z} = \iint (x\sigma_{xy} - y\sigma_{xx})dS \end{cases}$$

↪ Very complex. Additional assumptions are needed about how stresses are distributed over  $(S)$ .

# How to calculate the stress? (normal stress)

## ❖ Stress vector in the case of pure bending:

- No shear stresses on the cross-section

$$\begin{aligned}\varepsilon_{xx}(x, y) &= -y\kappa(x) \\ \Rightarrow \sigma_{xx}(x, y) &= -Ey\kappa(x)\end{aligned}$$

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## ❖ Stress-momentum relationship:

$$\mathbf{M} = \iint \mathbf{GM} \times \boldsymbol{\phi}(M, \mathbf{i}) \mathbf{k} dS = \iint (y\mathbf{j} + z\mathbf{k}) \times (-Ey\kappa(x)\mathbf{i}) dS$$

$$\Rightarrow \mathbf{M} = E\kappa(x) \left( \iint y^2 dS \mathbf{k} - \iint yz dS \mathbf{j} \right)$$

= 0, since the cross-section is symmetric to the  $z = 0$  plane.

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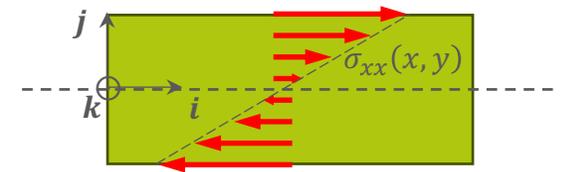
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$$\Rightarrow \mathbf{M} = E\kappa(x)I_x \mathbf{k} \Rightarrow B_{X,z}(x) = E\kappa(x)I_x$$

$I_x$  is the moment of inertia

$$\sigma_{xx}(x, y) = -\frac{B_{X,z}(x)}{\iint y^2 dS} y \Rightarrow \sigma_{xx}(x, y)_{max} = -\frac{B_{X,z}(x)}{I_x} y_{max}$$



# How to calculate the moment of inertia?

❖ Definition:

$$I_x = \iint y^2 dS$$

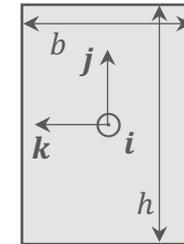
# How to calculate the moment of inertia?

❖ **Definition:**

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❖ **For a full rectangular beam:**

$$I_x = \iint y^2 dS = \iint y^2 dy dz$$
$$I_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy \times \int_{-\frac{b}{2}}^{\frac{b}{2}} dz = \frac{bh^3}{12}$$



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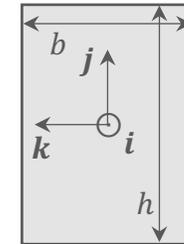
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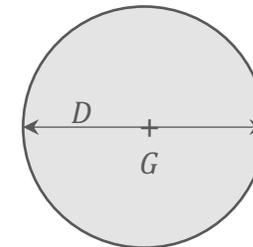
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$$I_x = \frac{\pi D^4}{64}$$



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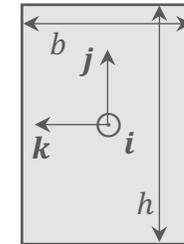
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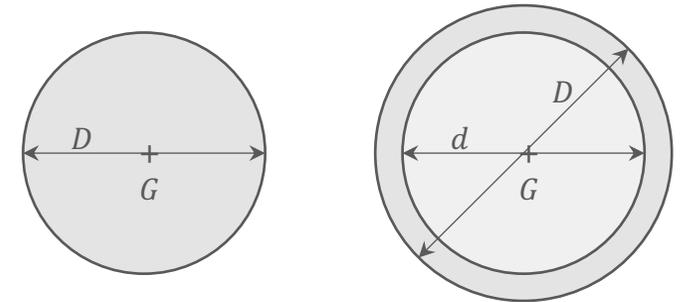
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## ❖ For a circular beam:

$$I_x = \frac{\pi D^4}{64}$$



## ❖ For a tube:

$$I_x = \frac{\pi(D^4 - d^4)}{64}$$

↳  $I_x$  depends on the cross-section shape. There are data sheets for complicated shapes.

# How to calculate the stress? (shear stress)

## ❖ Axial strain:

- Two adjacent fibers located at different  $y$  do not undergo the same axial strain ( $\varepsilon_{xx}$ ):

$$\varepsilon_{xx}(x, y) = -y\kappa(x)$$

⇒ It induces a relative longitudinal sliding between neighboring fibers.

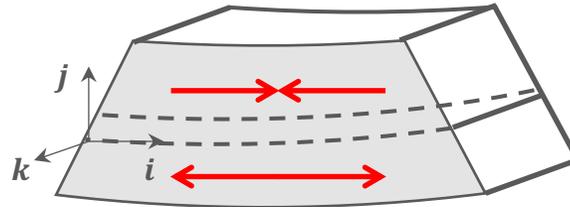
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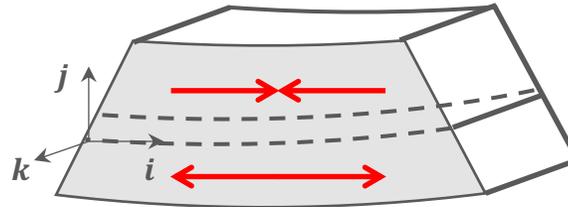
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- Shear stress ( $\tau_{xy}$ ) varies with the distance from the neutral axis ( $y$ ):  $\tau_{xy}(x, y)$ 
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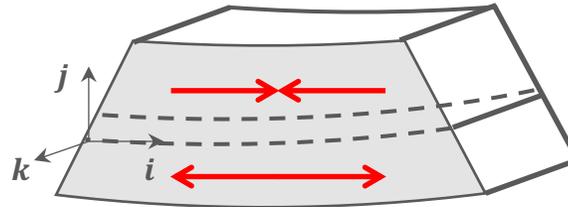
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$$\tau_{xy}^{avg}(x) = \frac{S_{X,y}(x)}{A_{xy}} \quad \text{where} \quad S_{X,y}(x) = -\frac{dB_{X,z}(x)}{dx}$$

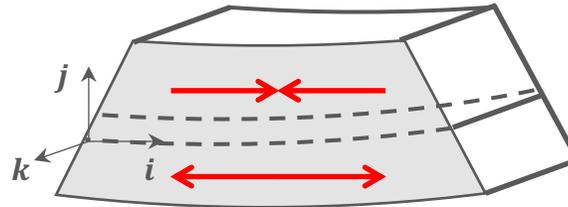
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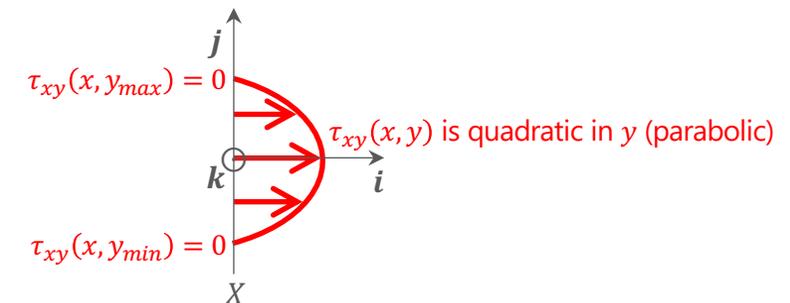
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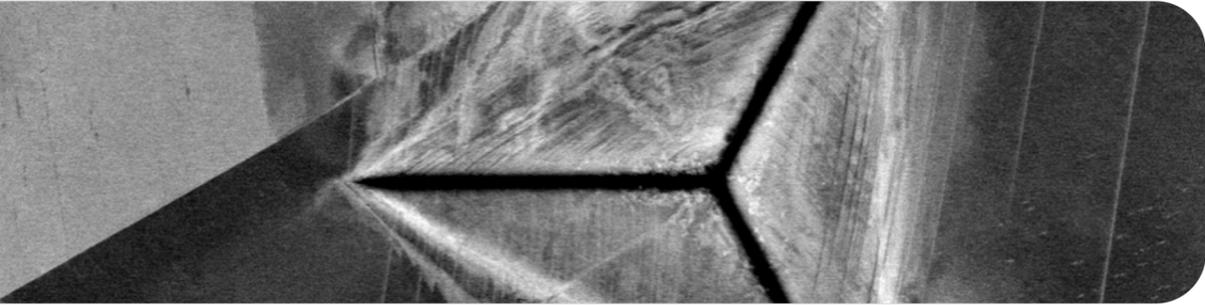
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- In the reality,  $\tau_{xy}(x, y)$  is non-uniform and must be computed locally.



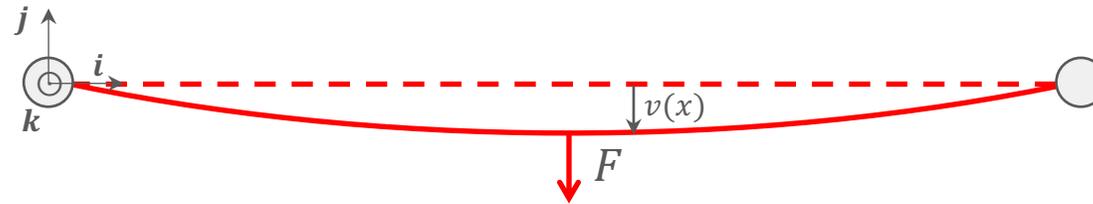


## Beam deflection analysis

# Deflection calculation

## ❖ Definition:

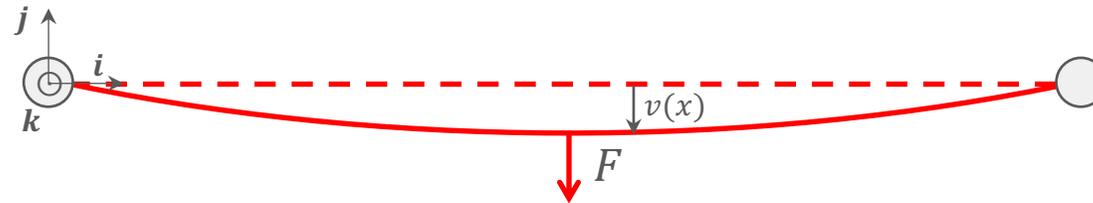
- The deflection is the transverse displacement of the beam axis along  $j$  at the position  $x$  under loading.



# Deflection calculation

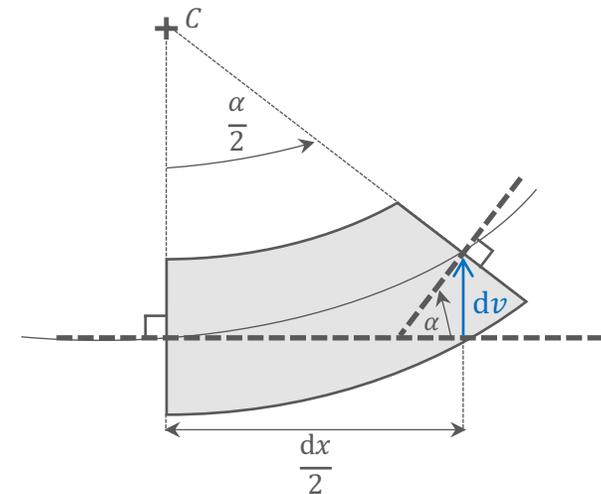
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## ❖ Calculation of the deflection $v(x)$ :

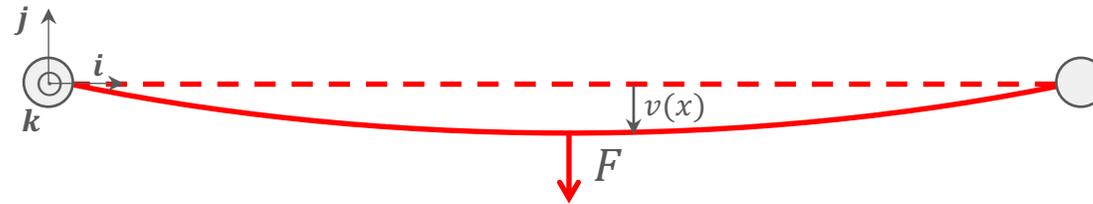
$$\tan \frac{\alpha(x)}{2} = \frac{dv(x)}{\frac{dx}{2}} \approx \alpha(x)$$



# Deflection calculation

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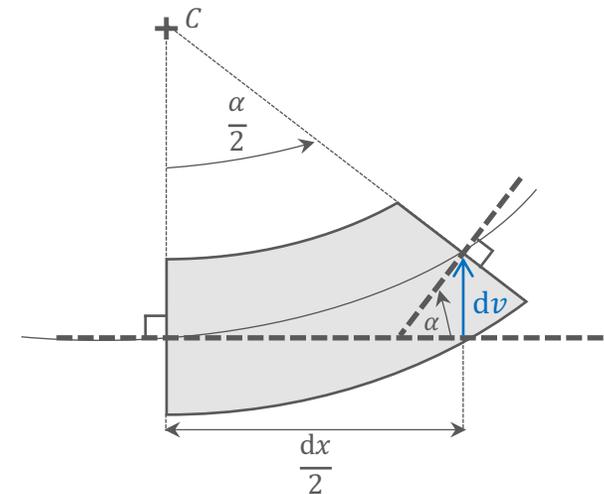
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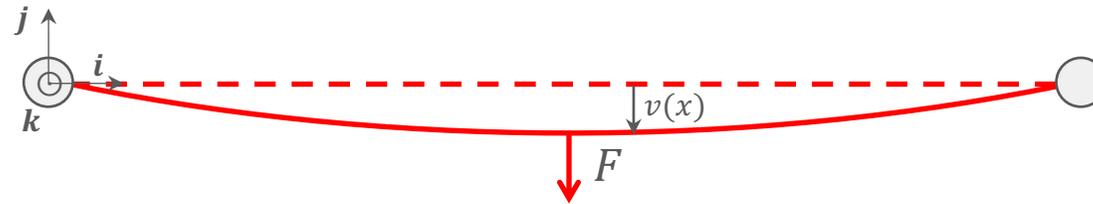
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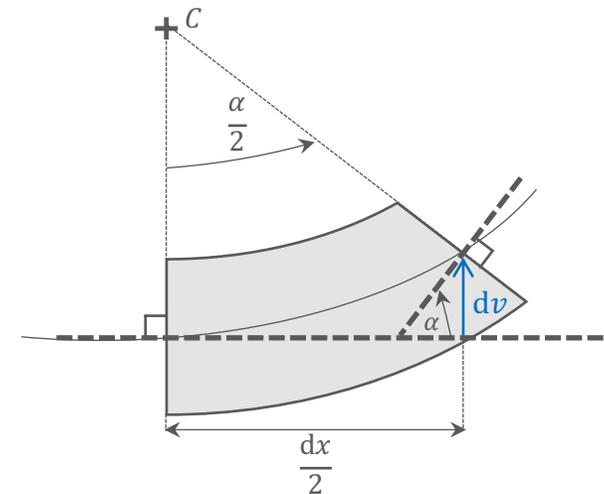
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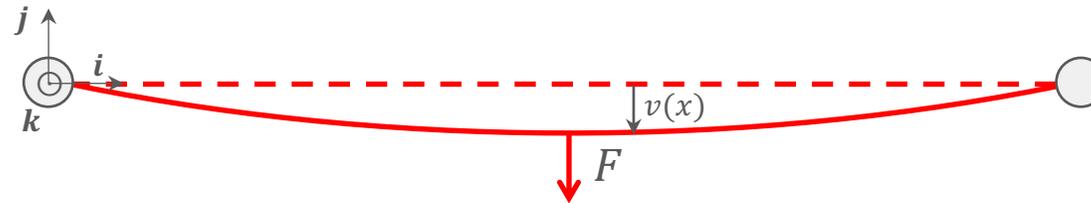
$$\Rightarrow \frac{d\alpha(x)}{dx} = \frac{d^2v(x)}{dx^2} \text{ but } \frac{d\alpha(x)}{dx} = \kappa(x) = \frac{B_{X,z}(x)}{EI_x}$$



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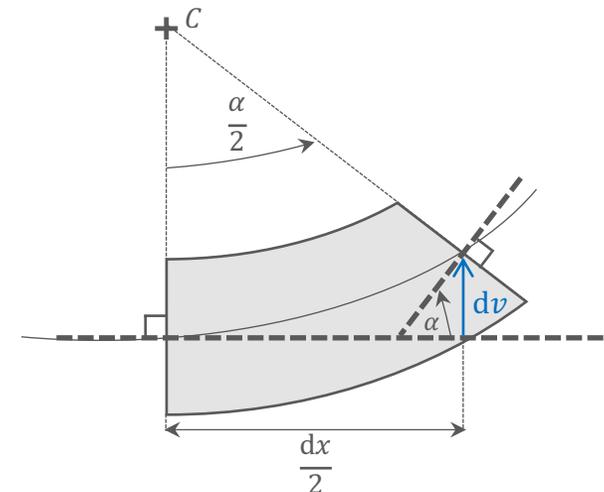
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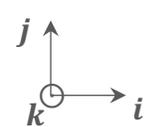
$$\Rightarrow \frac{d^2v(x)}{dx^2} = \frac{B_{X,Z}(x)}{EI_x}$$

- The deflection  $v(x)$  is then computed by successive integrations.



# Summary

Loadings	Deformation	Stress	Governing equations	Coefficients	Elasticity limit
Tension	$\varepsilon_{xx} > 0$	$\sigma_{xx} = \frac{N}{A_x} > 0$	$\sigma_{xx} = E\varepsilon_{xx}$ $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$	$E, \nu$	$R_e$
Compression	$\varepsilon_{xx} < 0$	$\sigma_{xx} = \frac{N}{A_x} < 0$	$\sigma = E\varepsilon_{xx}$ $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$	$E, \nu$	$R_e$
Shear	$\gamma_{xy}$	$\tau_{xy} = \frac{S_y}{A_x}$	$\tau_{xy} = G\gamma_{xy}$ $G = \frac{E}{2(1+\nu)}$	$G$	$\tau_e$
Torsion	$\theta$	$\tau, \sigma = 0$	$\tau = \frac{T}{I_G}\rho$ $\tau = G\theta\rho$	$G, I_G$	$\tau_e$
Bending	$\kappa$	$\sigma_{xx}, \tau_{xy}$	$\sigma_{xx}(x, y) = -\frac{B_z(x)}{I_x}y$ $\tau_{xy}^{avg}(x) = \frac{S_y(x)}{A_{xy}}$ $\frac{d^2v(x)}{dx^2} = \frac{B_z(x)}{EI_x}$	$E, I_x$	$R_e, \tau_e$





30

Thanks for your listening!

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