LABORATOIRE D'ÉTUDE DES MICROSTRUCTURES ET DE MÉCANIQUE DES MATÉRIAUX

UNIVERSITÉ DE LORRAINE

enim École Nationale d'Ingénieurs de Metz

Georgia Tech

COE–3001: Mechanics of deformable bodies

Chapter 4: torsion

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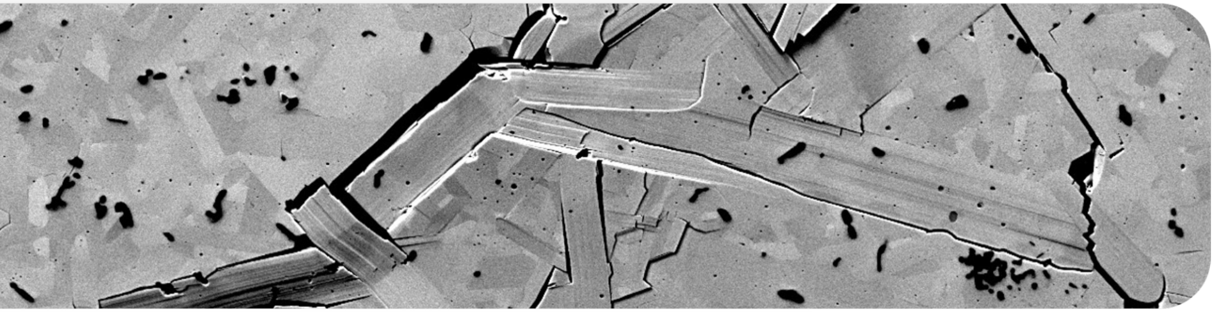

Deformation measurement

Deformation measure: twist rate

❖ **Reminder:**

$$\left\{\mathcal{T}_X^{coh,torsion}\right\} = \begin{Bmatrix} 0 & T_X \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}$$

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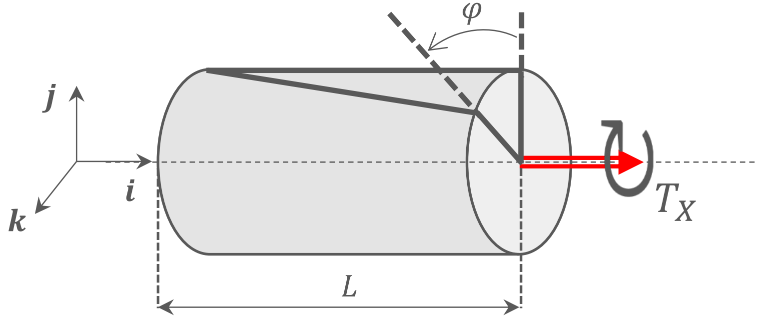
- For a shaft of length L, the angle of torsion (φ) corresponds to the rotation between the two ends of the shaft.



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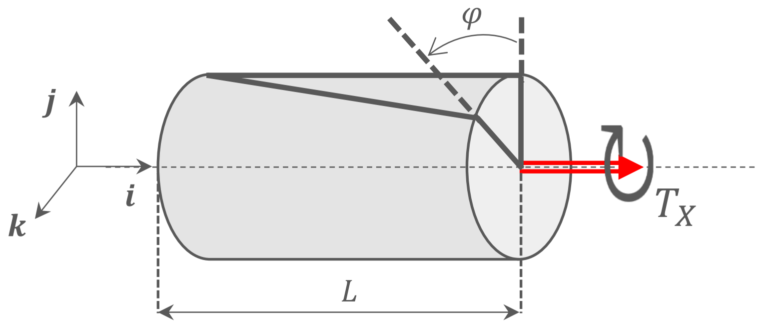

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- It depends on the shaft length (L).
- Even though the local deformation state is the same.

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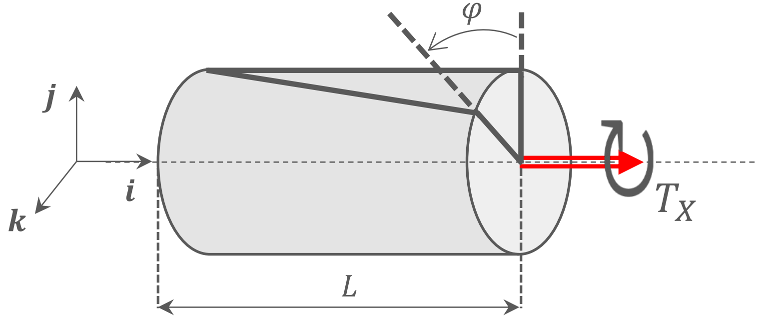


❖ Why the angle of torsion (φ) is not a suitable deformation measure?

- It depends on the shaft length (L).
- Even though the local deformation state is the same.

❖ Twist rate

- It is defined as the angle of torsion per unit length of the shaft:

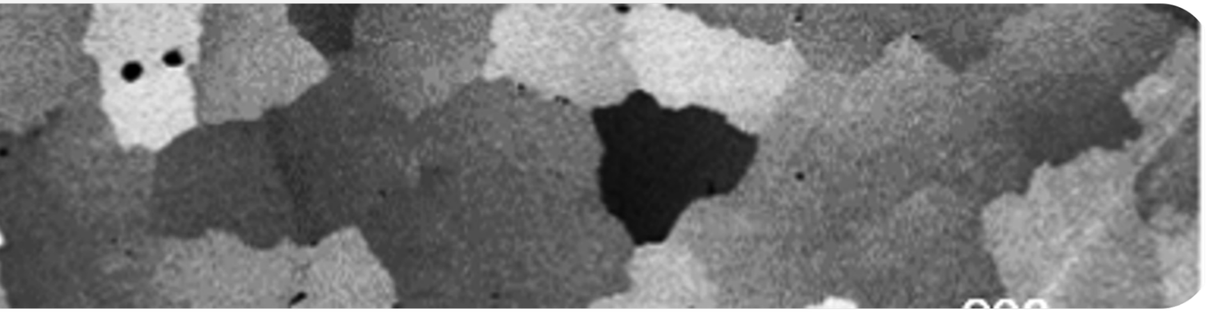
$$\theta(x) = \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}$$

- For uniform torsion ($\theta(x) = \theta$):

$$\theta = \frac{\mathrm{d}\varphi}{\mathrm{d}x} \Rightarrow \int_{\varphi_0=0}^{\varphi_L=\varphi} \mathrm{d}\varphi = \int_0^L \theta \mathrm{d}x$$

$$\Rightarrow \varphi - 0 = \theta(L-0) \Rightarrow \varphi = \theta L$$

$$\Rightarrow \theta = \frac{\varphi}{L}$$

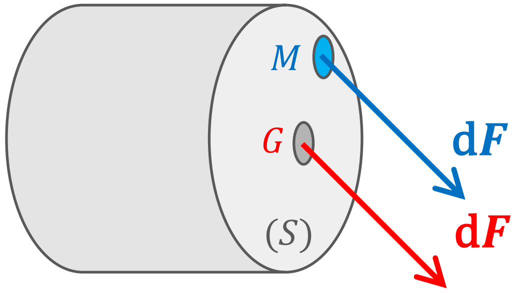


Stress

Relationship between ϕ and the internal forces

- **Cohesion torsor in M:**

$$\{\mathcal{T}_M^{coh}\} = \begin{Bmatrix} \boldsymbol{R}_M = \mathrm{d}\boldsymbol{F} \\ \boldsymbol{M}_M = \boldsymbol{0} \end{Bmatrix}$$



- **Let's transport this torsor into G near M:**

$$\{\mathrm{d}\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R}_G = \boldsymbol{R}_M = \mathrm{d}\boldsymbol{F} \\ \mathrm{d}\boldsymbol{M}_G = \boldsymbol{0} + \boldsymbol{GM}\times\boldsymbol{R}_M \end{Bmatrix} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R}_G = \mathrm{d}\boldsymbol{F} \\ \mathrm{d}\boldsymbol{M}_M = \boldsymbol{GM}\times\mathrm{d}\boldsymbol{F} \end{Bmatrix}$$

BUT $\boldsymbol{\phi}(M,\boldsymbol{n}) = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}S} \Rightarrow \mathrm{d}\boldsymbol{F} = \boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S$

$$\Rightarrow \{\mathrm{d}\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R}_G = \boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S \\ \mathrm{d}\boldsymbol{M}_G = \boldsymbol{GM}\times\boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S \end{Bmatrix}$$

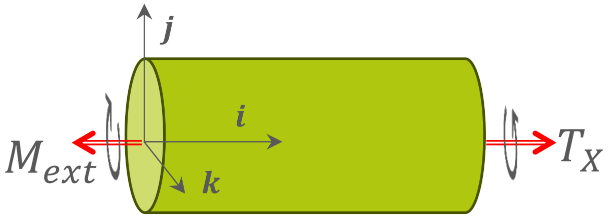
BUT $\boldsymbol{\phi}(M,\boldsymbol{n}=\boldsymbol{i}) = \sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k}$ AND $\boldsymbol{GM} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

- **After integration over (S):**

$$\{\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \boldsymbol{R}_G = \iint (\sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k})\mathrm{d}S \\ \boldsymbol{M}_G = \iint \boldsymbol{GM}\times(\sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k})\mathrm{d}S \end{Bmatrix} = \begin{Bmatrix} N_G = \iint \sigma_{xx}\mathrm{d}S & T_G = \iint (y\sigma_{xz} - z\sigma_{xy})\mathrm{d}S \\ S_{G,y} = \iint \sigma_{xy}\mathrm{d}S & B_{G,y} = \iint (z\sigma_{xz} - x\sigma_{xz})\mathrm{d}S \\ S_{G,z} = \iint \sigma_{xz}\mathrm{d}S & B_{G,z} = \iint (x\sigma_{xy} - y\sigma_{xx})\mathrm{d}S \end{Bmatrix}$$

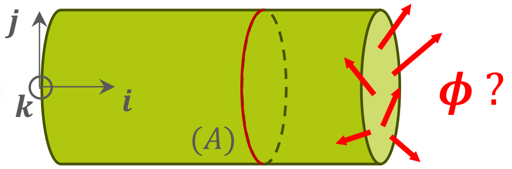
Very complex. Additional assumptions are needed about how stresses are distributed over (S).

How to calculate the stress?



Equilibrium ⇒ $M_{ext}\boldsymbol{i} = -T_X\boldsymbol{i}$

$$T_X = \iint (y\sigma_{xz} - z\sigma_{xy})\mathrm{d}S$$



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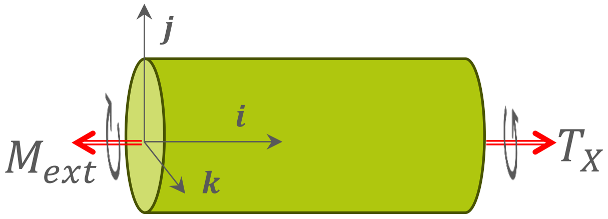

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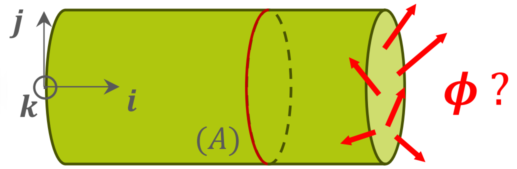
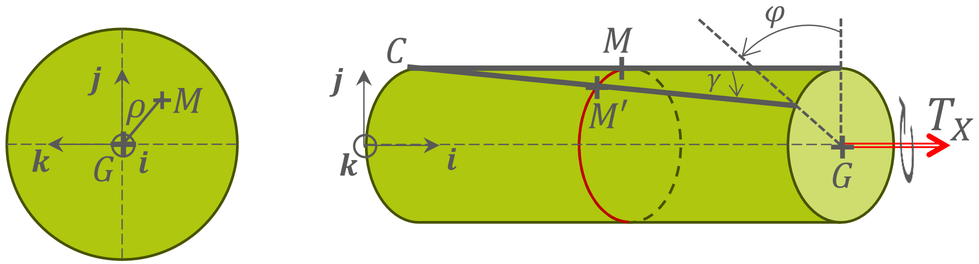


Equilibrium ⇒ $M_{ext}\boldsymbol{i} = -T_X\boldsymbol{i}$

❖ **We cannot assume that ϕ is constant.**

- After deformation, the point M located at a distance ρ from G, moves to M'.
- The line (CM') is distorted relative to the line (CM). This angular distortion defines the shear strain γ.
- For small deformations, γ is equal to the angular distortion between two initially parallel material fibers.



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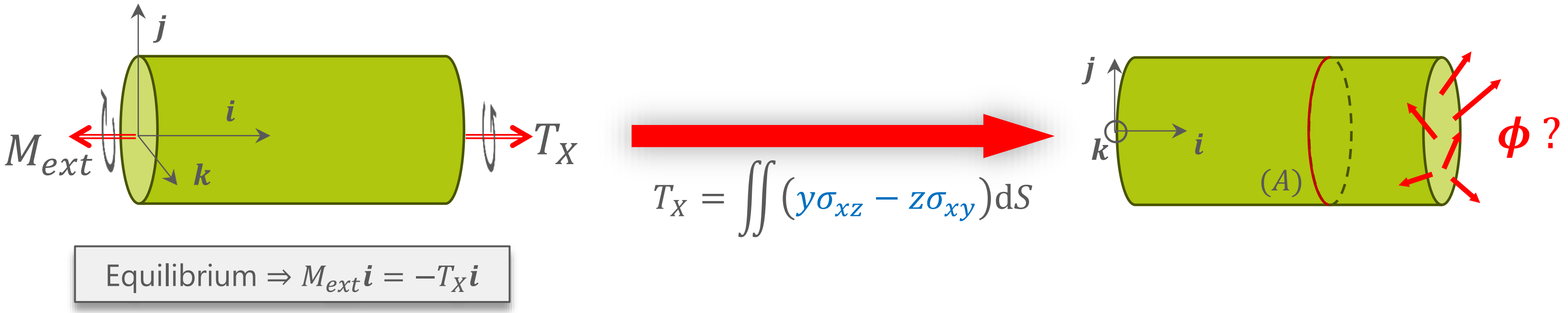

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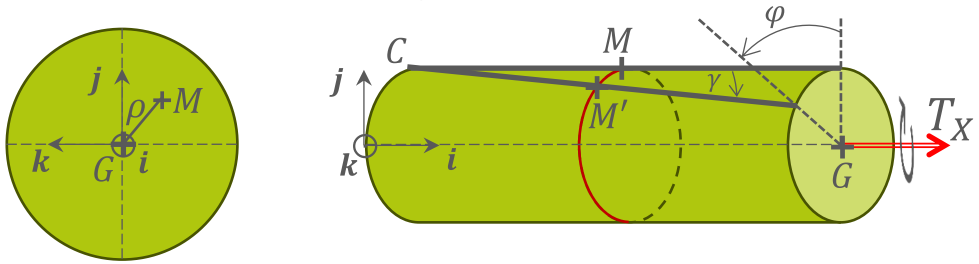

- Since the length along $\boldsymbol{i}$ remains unchanged, the axial strain is zero: $\varepsilon_{xx} = 0$

↳ $\boldsymbol{\phi}$ has no normal component. $\boldsymbol{\phi}(M, \boldsymbol{n} = \boldsymbol{i}) = \tau\boldsymbol{t}$.

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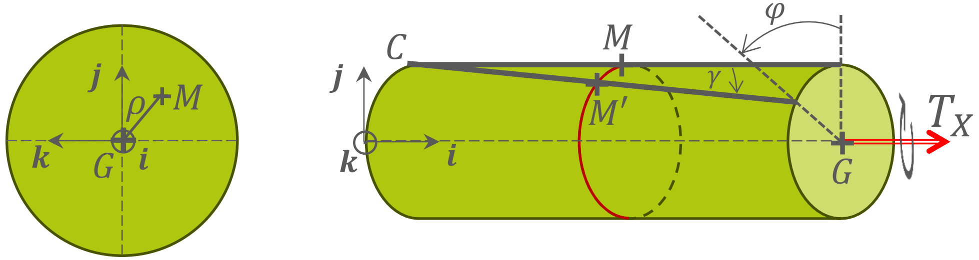

❖ **Stress-twist rate relationship:**

$$\tan\varphi = \frac{MM'}{\rho} \approx \varphi \Rightarrow \varphi\rho = MM' \text{ and } \tan\gamma = \frac{MM'}{L} \approx \gamma$$

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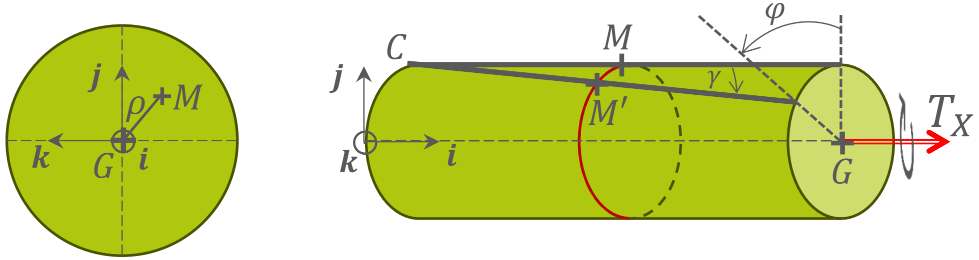

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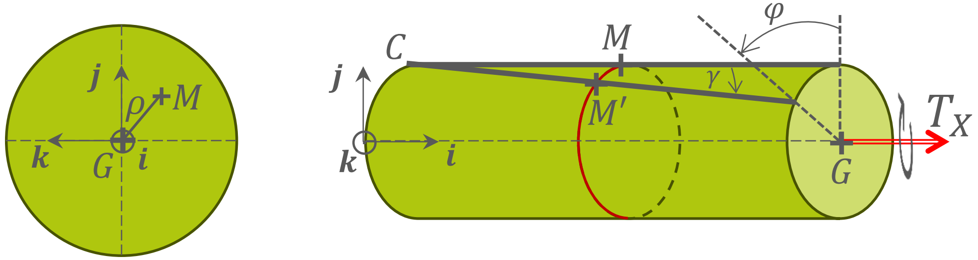

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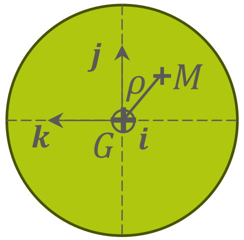

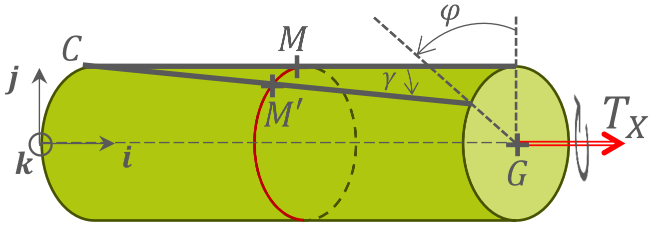

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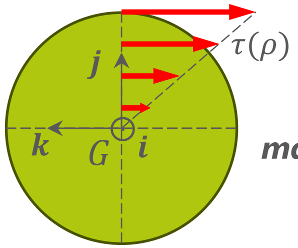
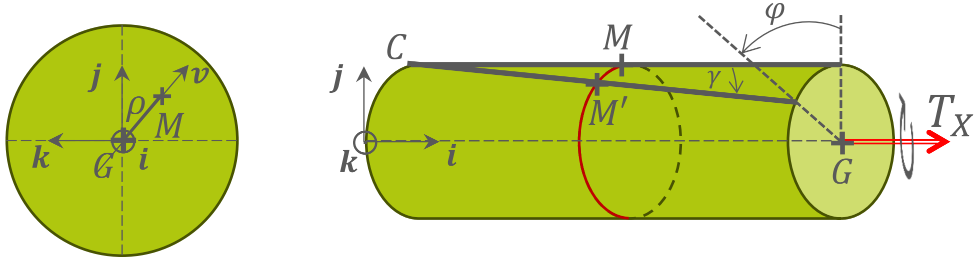
$$\tau = G\theta\rho$$



The shear stress (τ) is maximum at the outer surface.

How to calculate the stress?



❖ **Stress-torque relationship:**

The stress vector: $\boldsymbol{\phi}(M,\boldsymbol{i}) = \tau\boldsymbol{t} = G\theta\rho\boldsymbol{t}$

The torque: $\boldsymbol{M}_{ext} = M_{ext}\boldsymbol{i}$

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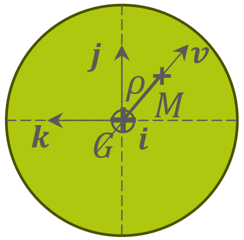

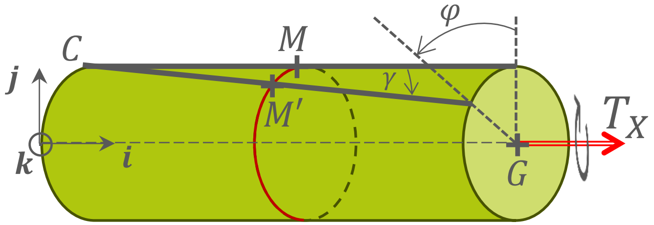

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$$\boldsymbol{M}_{ext} = \iint \boldsymbol{GM} \times \boldsymbol{\phi}(M, \boldsymbol{i})\mathrm{d}S = \iint \rho\boldsymbol{v} \times G\theta\rho\boldsymbol{t}\mathrm{d}S = \iint G\theta\rho^2\boldsymbol{v} \times \boldsymbol{t}\mathrm{d}S = \iint G\theta\rho^2\boldsymbol{i}\mathrm{d}S$$

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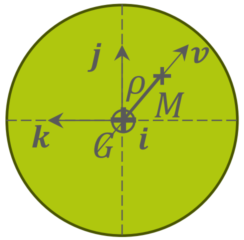

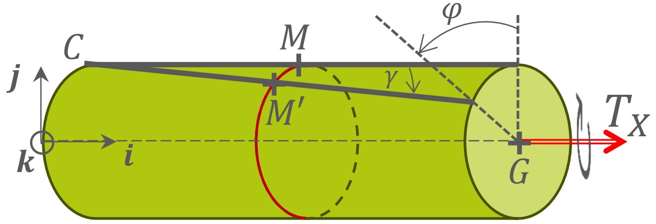

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$$\Rightarrow \boldsymbol{M_{ext}} = G\theta \iint \rho^2\mathrm{d}S\,\boldsymbol{i}$$

$$\tau = \frac{M_{ext}}{\iint \rho^2 \mathrm{d}s}\rho = \frac{M_{ext}}{I_G}\rho \Rightarrow \tau_{max} = \frac{M_{ext}}{I_G}R$$

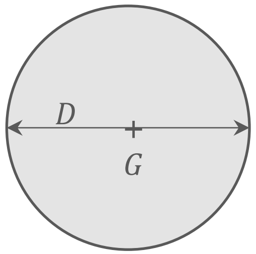
I_G is the polar moment of inertia

How to calculate the polar moment of inertia?

- **Definition:**

$$I_G = \iint \rho^2 \mathrm{d}S$$

- **For a full shaft:**



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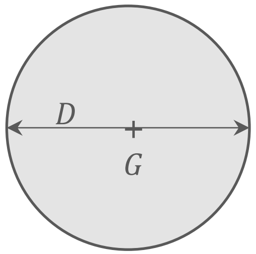
$$I_G = \int_0^{\frac{D}{2}} \rho^3 \mathrm{d}\rho \times \int_0^{2\pi} \mathrm{d}\theta = \frac{\pi D^4}{32}$$



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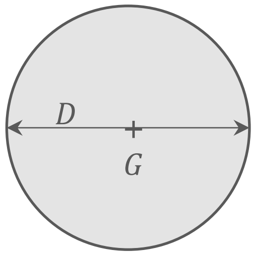
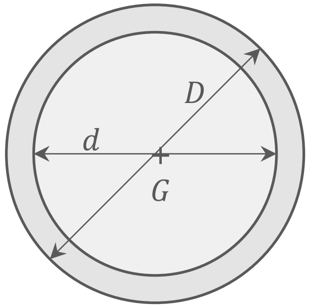


- **For a tube:**

$$I_G = I_G^D - I_G^d = \frac{\pi D^4}{32} - \frac{\pi d^4}{32}$$

$$I_G = \frac{\pi (D^4 - d^4)}{32}$$



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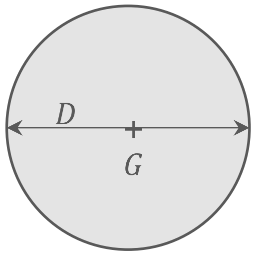

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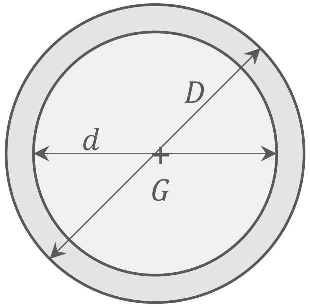
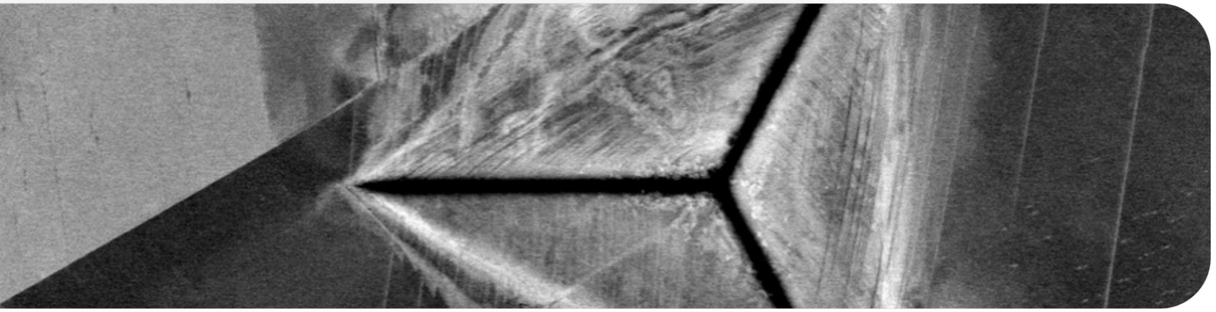


↳ I_G depends on the cross-section shape. There are data sheets for complicated shapes.



Torsion testing

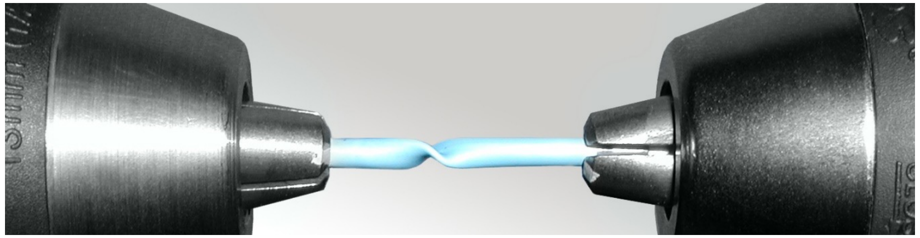
Principle of torsion test

❖ **Objective:**

- Characterize the material response under torsional loading
- Determine the shear stress distribution and the maximum shear stress
- Determine the torsional strength and failure torque

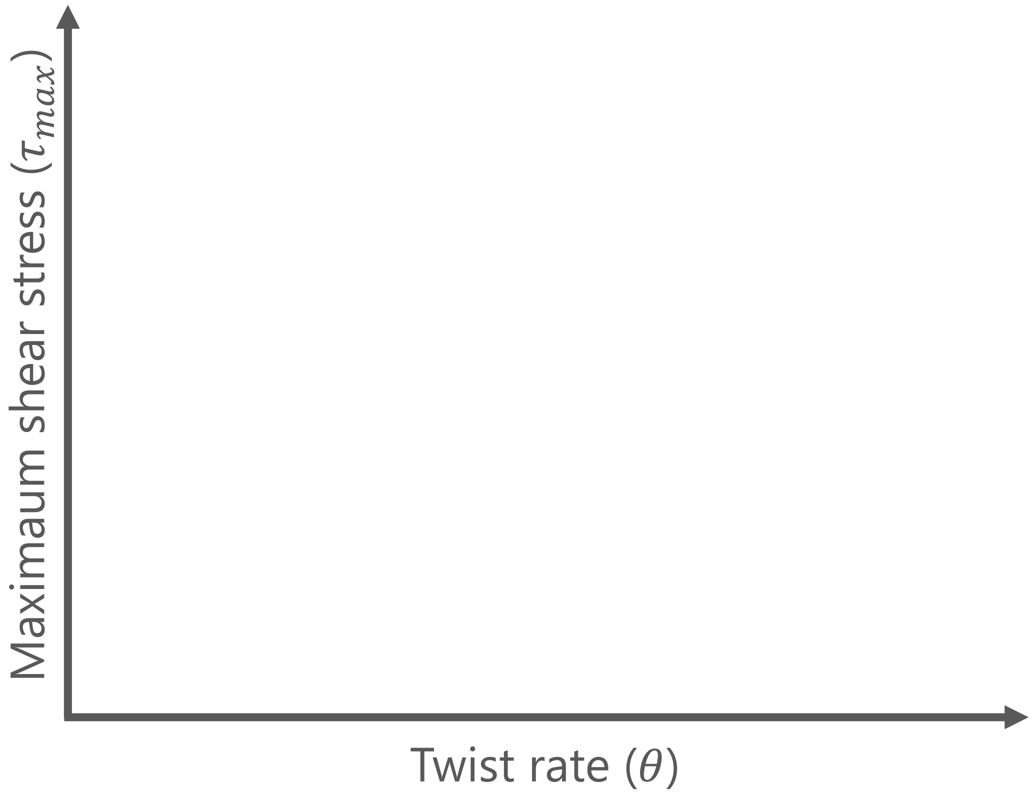
❖ **Principle:**

- Application of a torque (twisting moment) about the longitudinal axis of the specimen
- Relative rotation of cross-sections along the shaft
- Induces shear stresses that vary with the radial position
- Ideally produces pure torsional deformation (pure shear state)



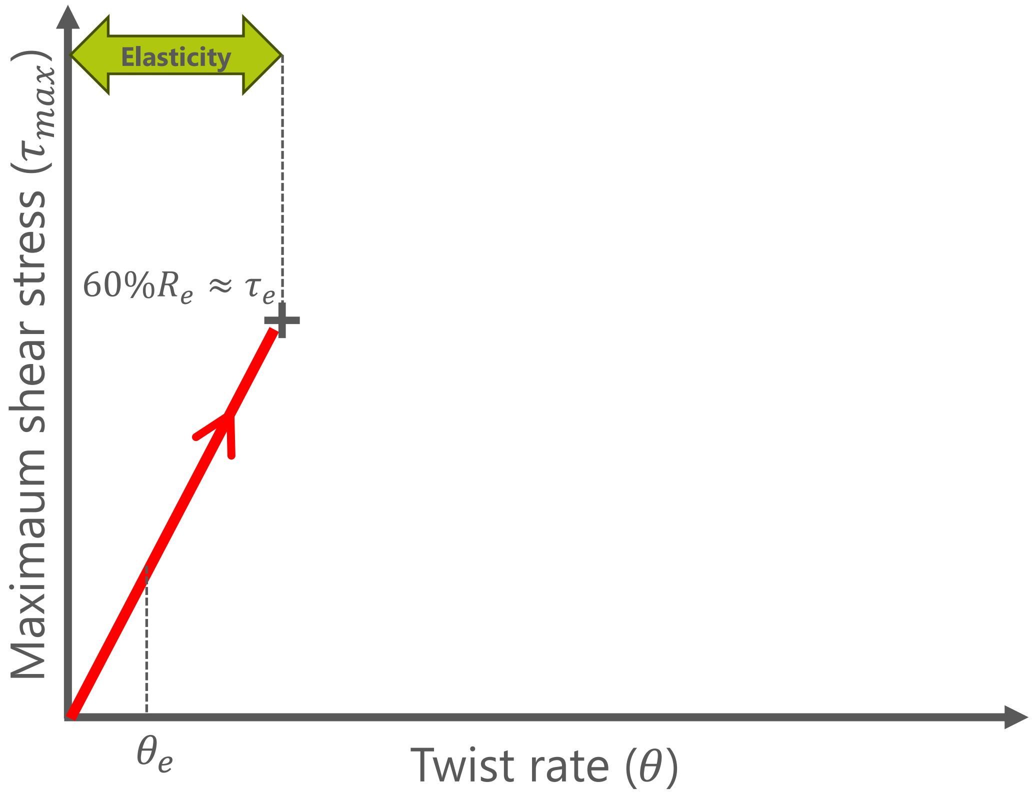
Instron

Use of results



- **Engineering shear stress-twist rate curve (ductile material):**
  - The vertical axis represents the engineering shear stress (τ), evaluated at the outer surface of the shaft.
  $$\tau_{max} = \frac{M_{applied}}{I_G} R$$
  - The horizontal axis represents the twist rate (θ), defined using the initial geometry.
  $$\theta = \frac{\varphi}{L}$$

Use of results



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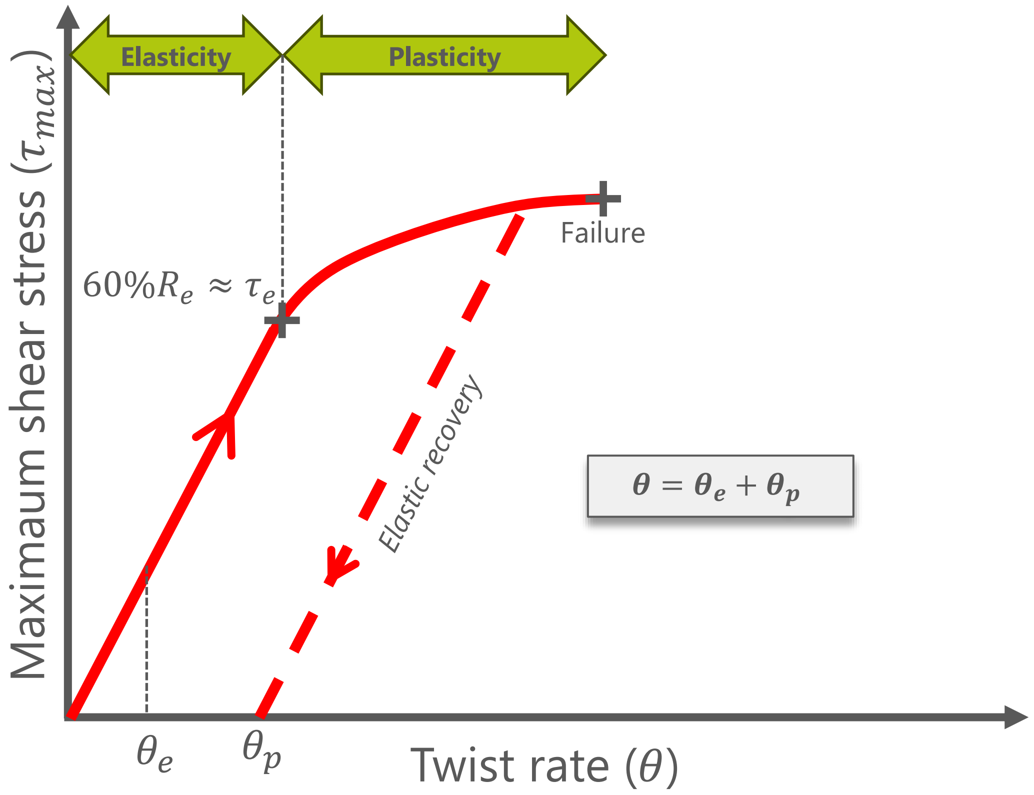
- **Elastic region ($\tau < \tau_e \approx 60\% R_e$):**
  - The initial linear part corresponds to the elastic regime.
  - Shear stress and strain are proportional (Hooke's law):
    $$\tau = G\gamma_e = G\rho\theta_e$$
  - Deformation is fully reversible upon unloading.
  - The slope of the torque-twist curve is given by: GI_G.
  - The elastic limit is reached at the yield stress, τ_e.

Use of results



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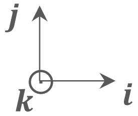
  - The elastic limit is reached at the yield stress, τ_e.
- **Plastic region ($\boldsymbol{\tau > \tau_e \approx 60\%R_e}$):**
  - Deformation becomes irreversible.
  - Plasticity initiates at the outer surface of the shaft.
  - The plastic zone progressively propagates towards the center.
  - Unloading from this region follows a linear elastic path.
  - After unloading, a permanent plastic twist (θ_p) remains.
  - Torsion allows significant plastic deformation before failure.

Summary

Loadings	Deformation	Stress	Governing equations	Coefficients	Elasticity limit
Tension	$\varepsilon_{xx} > 0$	$\sigma_{xx} = \frac{N}{A_x} > 0$	$\sigma_{xx} = E\varepsilon_{xx}$ $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$	E, ν	R_e
Compression	$\varepsilon_{xx} < 0$	$\sigma_{xx} = \frac{N}{A_x} < 0$	$\sigma = E\varepsilon_{xx}$ $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$	E, ν	R_e
Shear	γ_{xy}	$\tau_{xy} = \frac{S_y}{A_x}$	$\tau_{xy} = G\gamma_{xy}$ $G = \frac{E}{2(1+\nu)}$	G	τ_e
Torsion	θ	$\tau, \sigma = 0$	$\tau = \frac{T}{I_G}\rho$ $\tau = G\theta\rho$	G, I_G	τ_e



Thanks for your listening!

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