COE–3001: Mechanics of deformable bodies

Chapter 2: traction and compression

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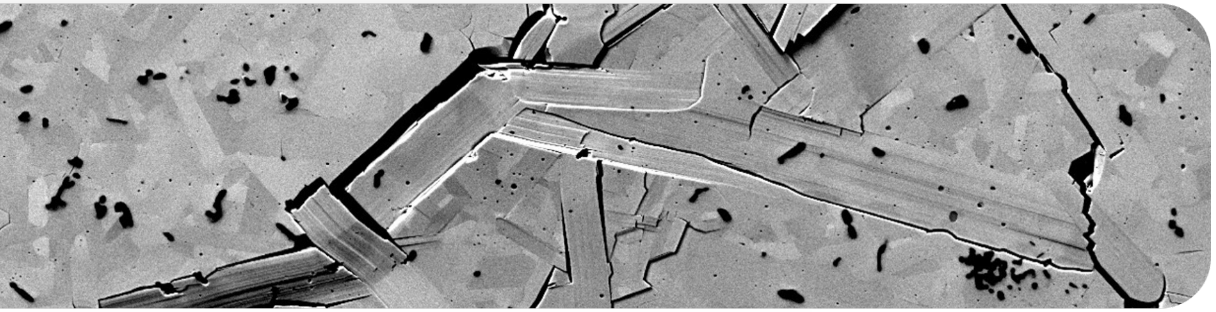

Deformation measurement

Deformation measure: strain

❖ **Reminder:**

$$\{\mathcal{T}_X^{coh,tract}\} = \begin{Bmatrix} N_X(>0) & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix};\ \{\mathcal{T}_X^{coh,comp}\} = \begin{Bmatrix} N_X(<0) & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}$$

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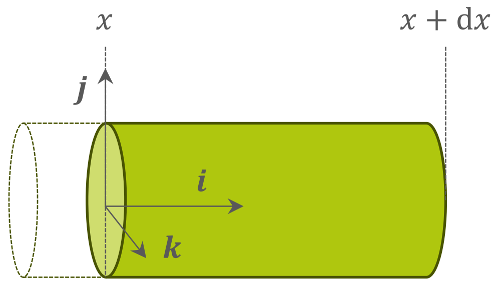

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- The coordinate x labels the initial reference position.

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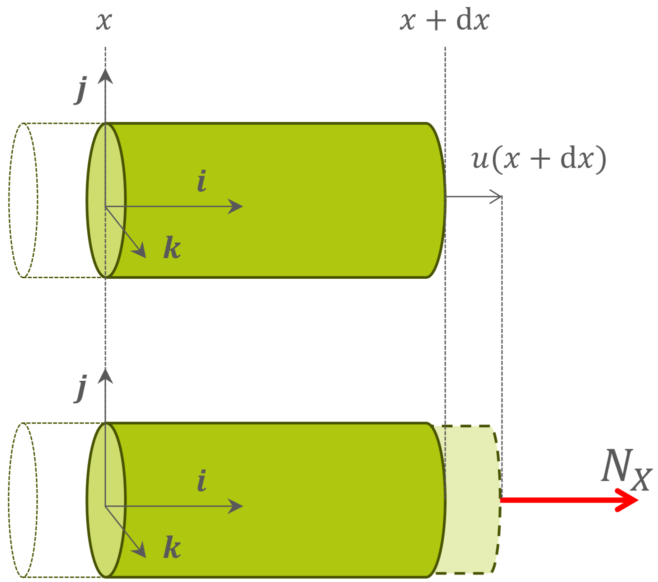
❖ **Strain:**

- The coordinate x labels the initial reference position.
- After loading, this point moves to the position $x + u(x)$ where $u(x)$ is the axial displacement field.
- A neighboring point initially located at $x + \mathrm{d}x$ moves to $x + \mathrm{d}x + u(x + \mathrm{d}x)$



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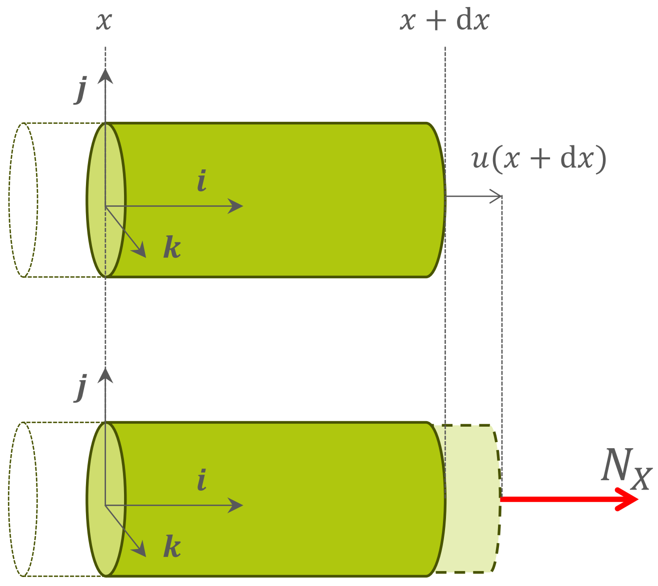
- The elongation of the infinitesimal element is given by:
  $$\mathrm{d}\ell = u(x + \mathrm{d}x) - u(x)$$
- The axial strain is defined as the relative elongation per unit length:
  $$\varepsilon_{xx} = \frac{\mathrm{d}\ell}{\mathrm{d}x}$$
- In the limit $dx \to 0$, this leads to the local strain definition:
  $$\varepsilon_{xx} = \frac{\partial \ell}{\partial x}$$



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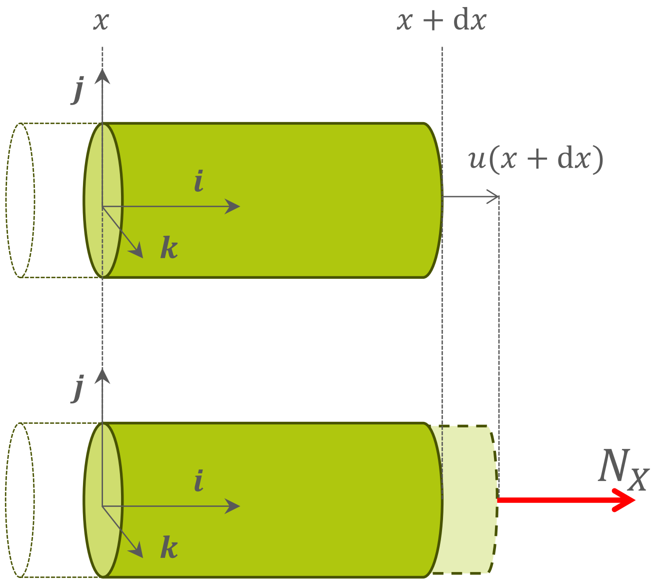
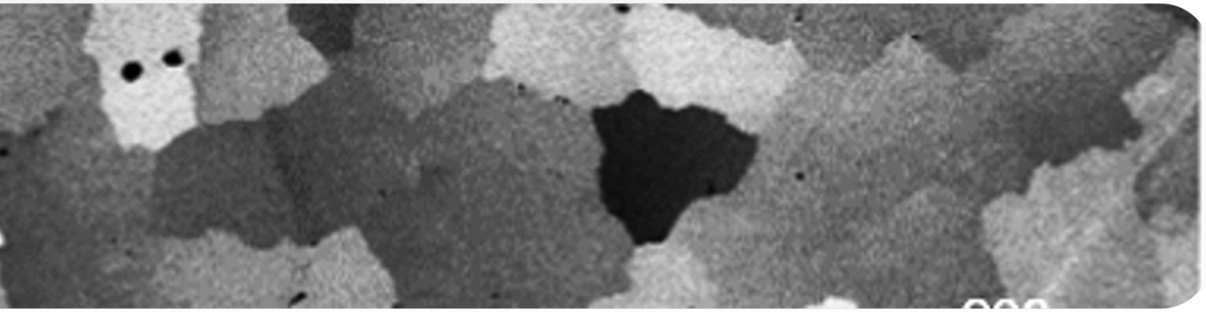


- If $N_x > 0 \Rightarrow \varepsilon_{xx} > 0 \Rightarrow$ tensile strain
- If $N_x < 0 \Rightarrow \varepsilon_{xx} < 0 \Rightarrow$ compressive strain
- Dimensionless (ratio of two lengths)

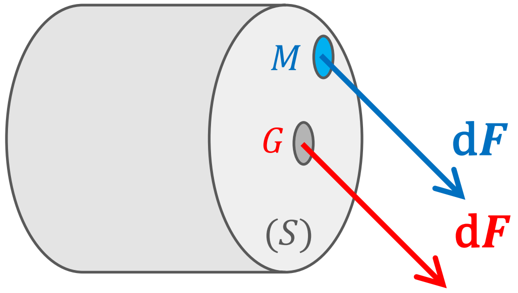


Stress

Relationship between ϕ and the internal forces

- **Cohesion torsor in M:**

$$\{\mathcal{T}_M^{coh}\} = \begin{Bmatrix} \boldsymbol{R}_M = \mathrm{d}\boldsymbol{F} \\ \boldsymbol{M}_M = \boldsymbol{0} \end{Bmatrix}$$



- **Let's transport this torsor into G near M:**

$$\{\mathrm{d}\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R}_G = \boldsymbol{R}_M = \mathrm{d}\boldsymbol{F} \\ \mathrm{d}\boldsymbol{M}_G = \boldsymbol{0} + \boldsymbol{GM} \times \boldsymbol{R}_M \end{Bmatrix} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R}_G = \mathrm{d}\boldsymbol{F} \\ \mathrm{d}\boldsymbol{M}_M = \boldsymbol{GM} \times \mathrm{d}\boldsymbol{F} \end{Bmatrix}$$

BUT $\boldsymbol{\phi}(M,\boldsymbol{n}) = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}S} \Rightarrow \mathrm{d}\boldsymbol{F} = \boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S$

$$\Rightarrow \{\mathrm{d}\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R}_G = \boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S \\ \mathrm{d}\boldsymbol{M}_G = \boldsymbol{GM} \times \boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S \end{Bmatrix}$$

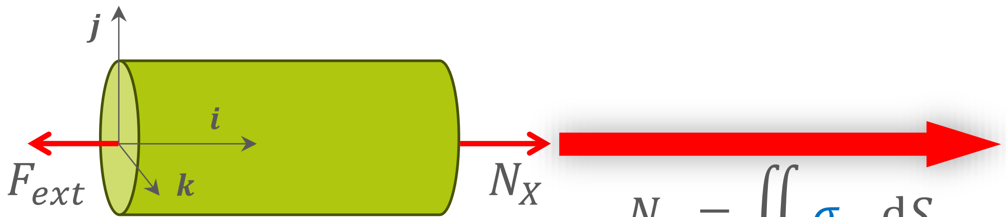
BUT $\boldsymbol{\phi}(M,\boldsymbol{n}=\boldsymbol{i}) = \sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k}$ AND $\boldsymbol{GM} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

- **After integration over (S):**

$$\{\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \boldsymbol{R}_G = \iint (\sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k})\mathrm{d}S \\ \boldsymbol{M}_G = \iint \boldsymbol{GM} \times (\sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k})\mathrm{d}S \end{Bmatrix} = \begin{Bmatrix} N_G = \iint \sigma_{xx}\mathrm{d}S & T_G = \iint (y\sigma_{xz} - z\sigma_{xy})\mathrm{d}S \\ S_{G,y} = \iint \sigma_{xy}\mathrm{d}S & B_{G,y} = \iint (z\sigma_{xz} - x\sigma_{xz})\mathrm{d}S \\ S_{G,z} = \iint \sigma_{xz}\mathrm{d}S & B_{G,z} = \iint (x\sigma_{xy} - y\sigma_{xx})\mathrm{d}S \end{Bmatrix}$$

➯ Very complex. Additional assumptions are needed about how stresses are distributed over (S).

How to calculate the stress?



$$N_X = \iint \sigma_{xx} \mathrm{d}S$$

Equilibrium $\Rightarrow F_{ext}\boldsymbol{i} = -N_X\boldsymbol{i}$

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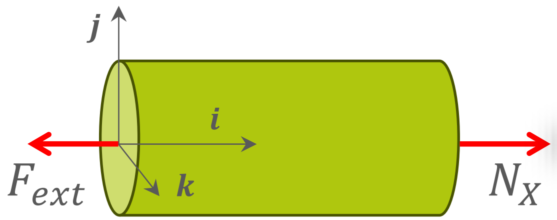

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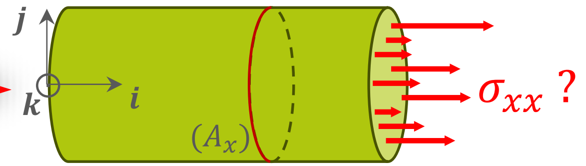

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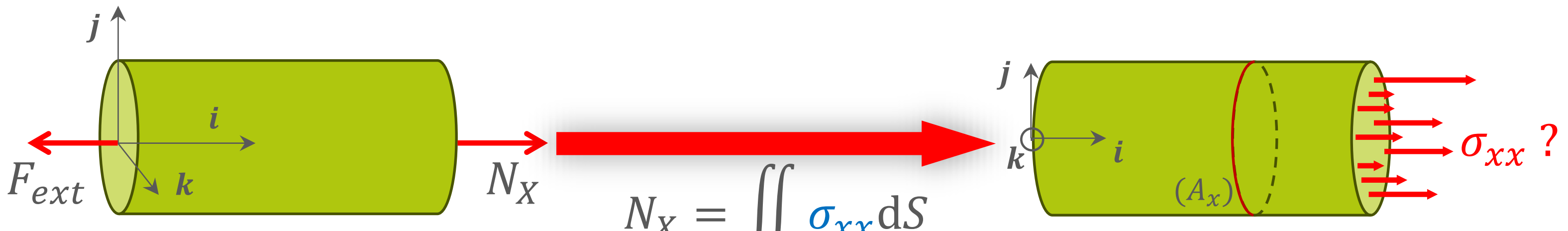

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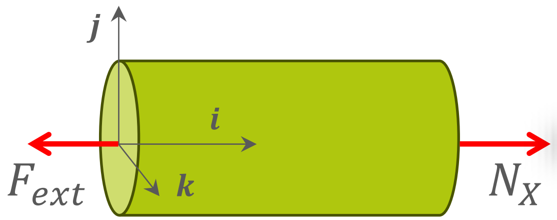
- **We assume that σ_{xx} is constant.**
  - Uniform cross-section: the area (A) does not vary along x.
  - Constant axial force: the normal force F_{ext} is the same at every cross-section (no distributed axial loads).
  - Uniaxial loading: only axial tension is applied (no bending, no shear).
  - Homogeneous material: material properties are identical everywhere.
  - Small deformations: geometry changes are negligible.

How to calculate the stress?



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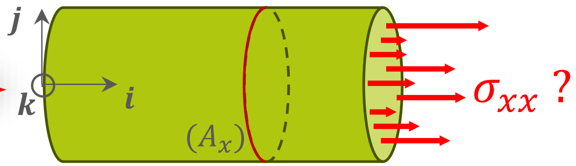

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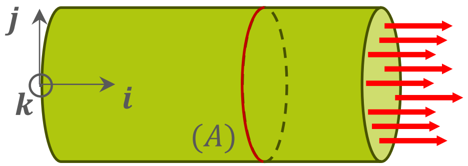
$$N_X = \iint \sigma_{xx} \mathrm{d}S \Leftrightarrow F_{ext} = \iint \sigma_{xx} \mathrm{d}S = \sigma_{xx} \iint \mathrm{d}S = \sigma_{xx} A_x$$

$$\Leftrightarrow \sigma_{xx} = \frac{F_{ext}}{A_x}$$



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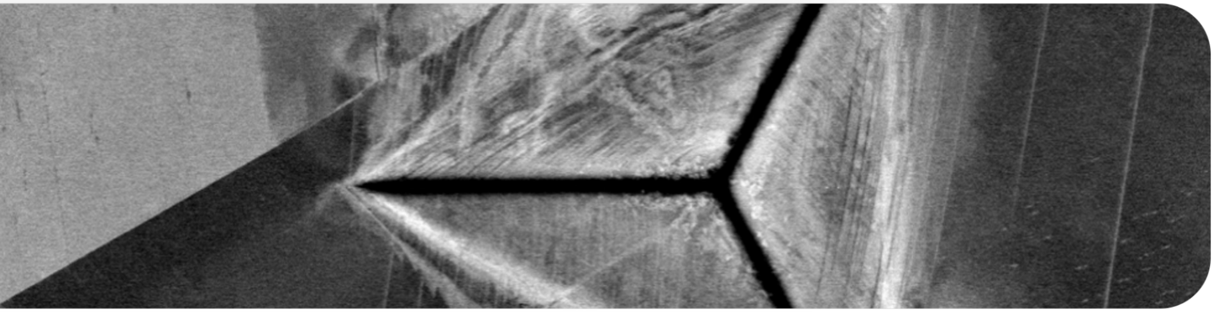
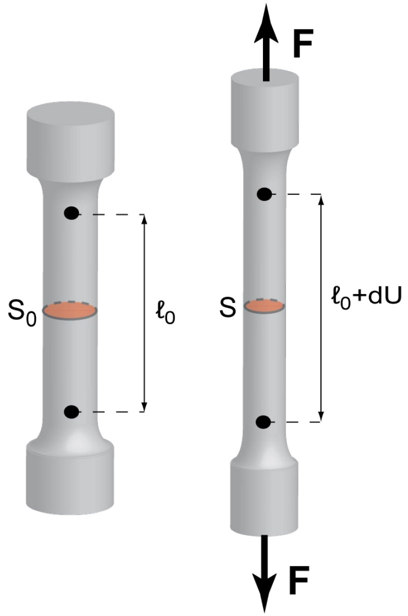


Tensile and compression testing

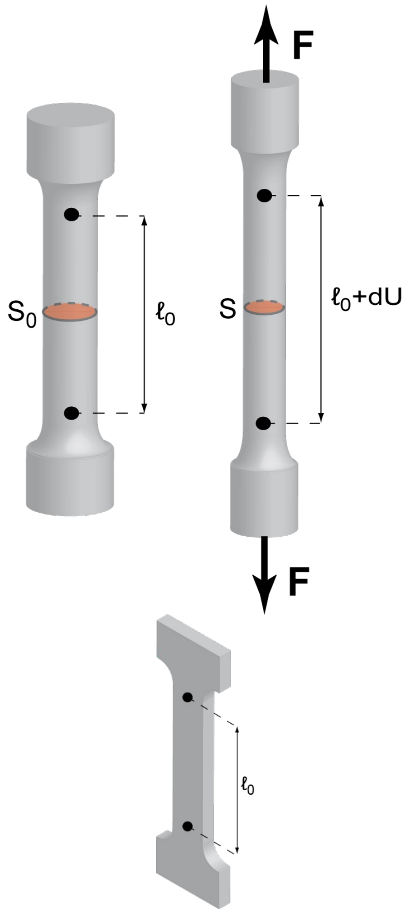
The specimens and principle



❖ **Principle of uniaxial tensile testing:**

- Most common test
- Standardized specimen geometry

The specimens and principle



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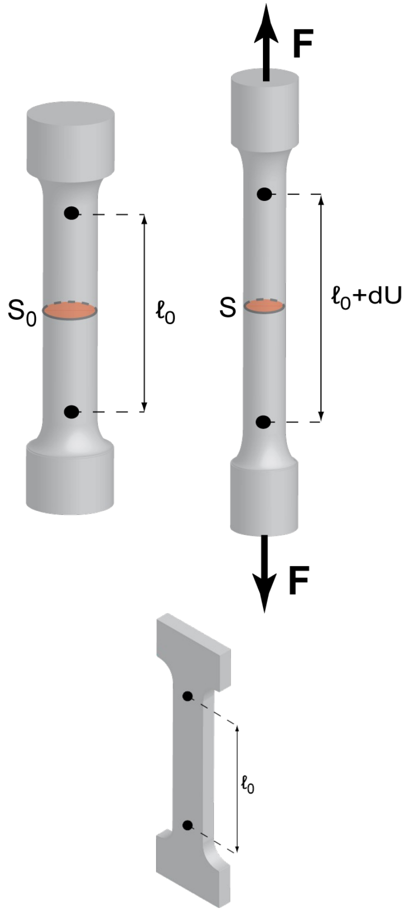

❖ Principle of uniaxial tensile testing:

- Most common test
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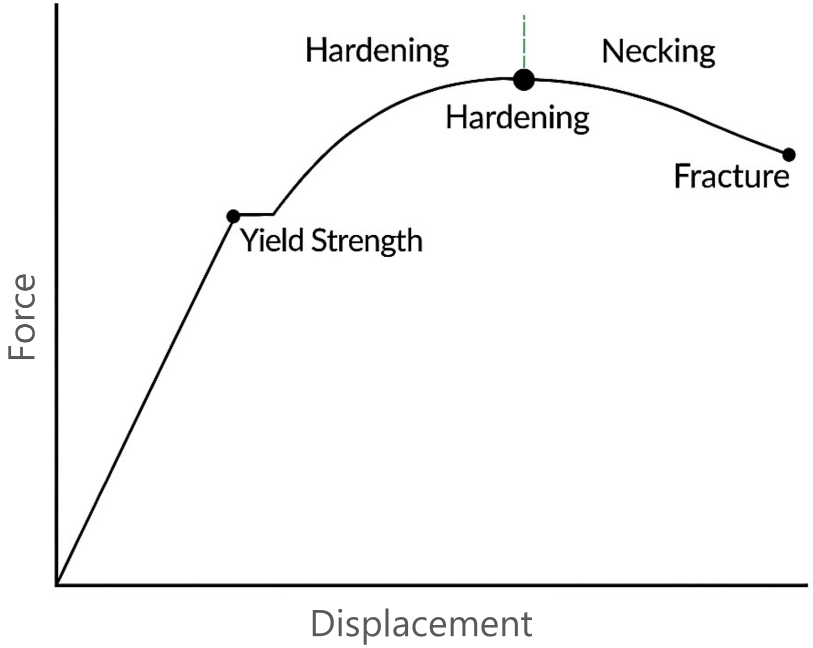
- The displacement U and the applied force F are measured



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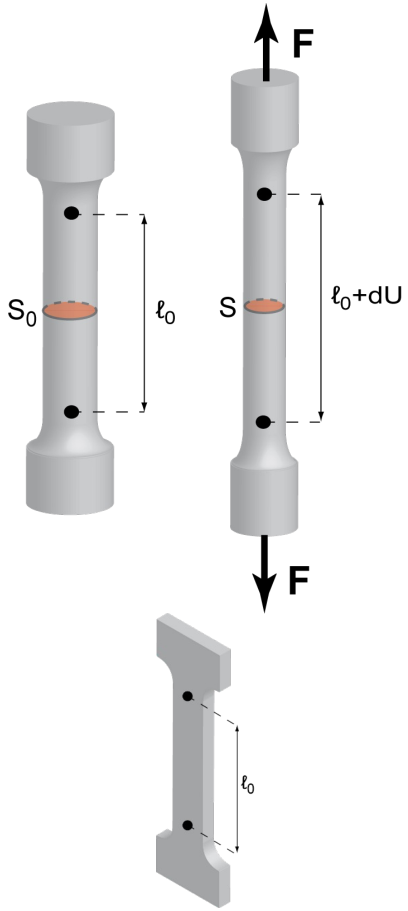

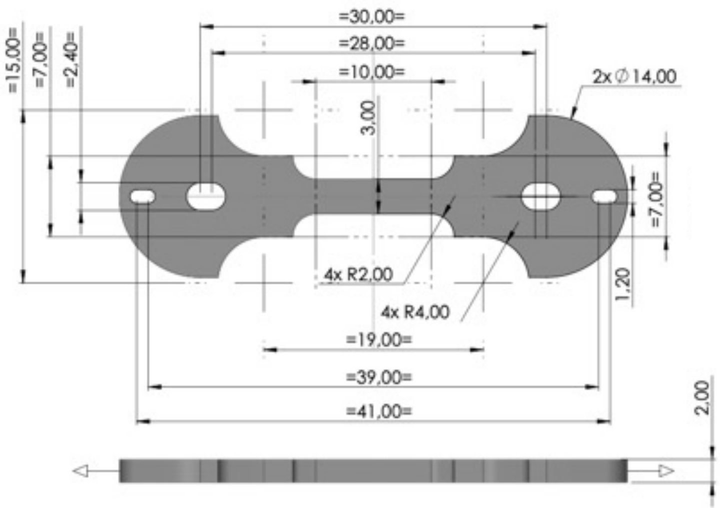

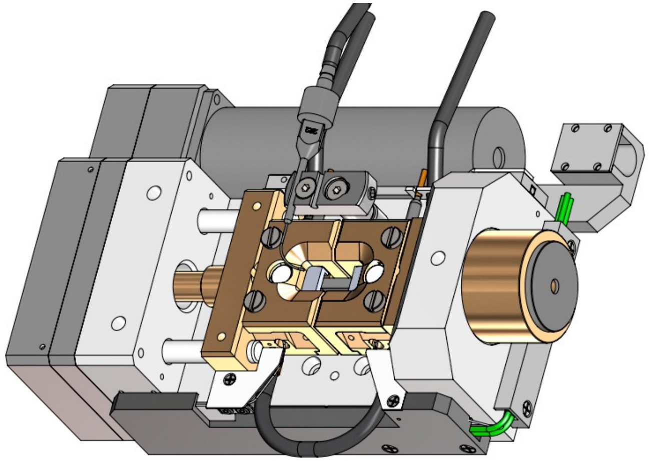

MT1000 (NewTec)

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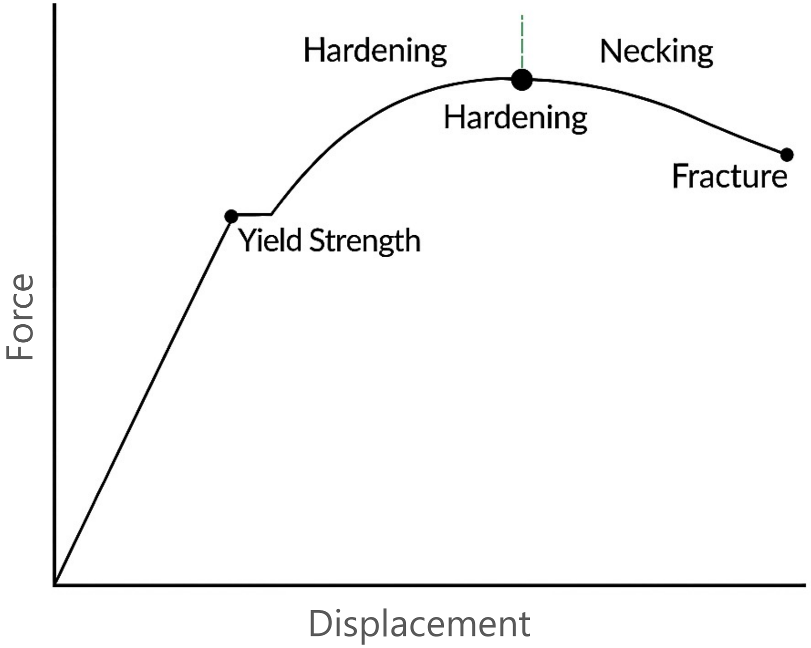
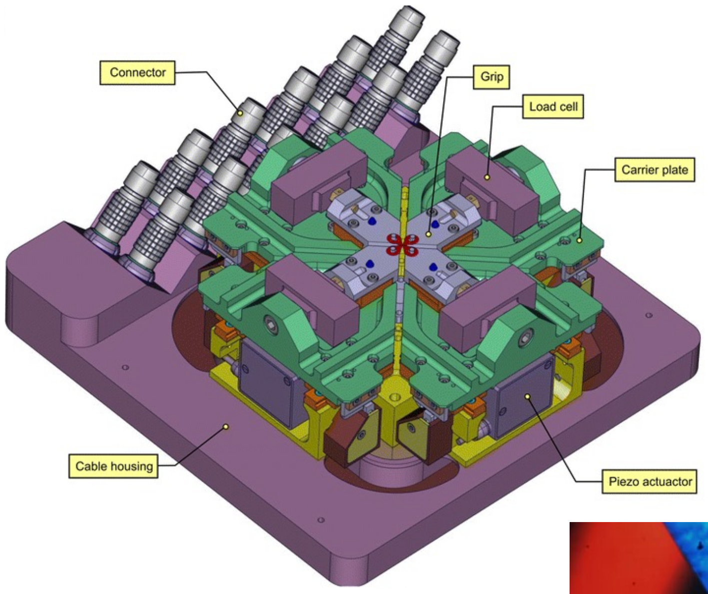
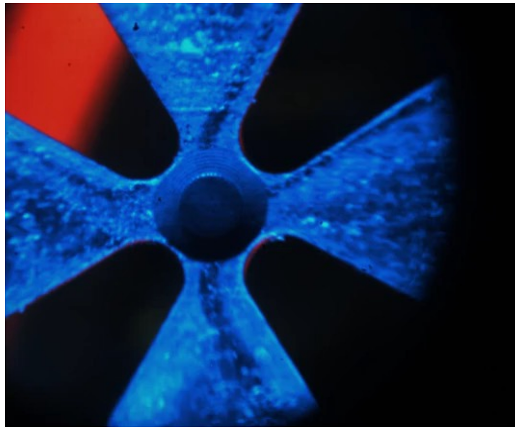
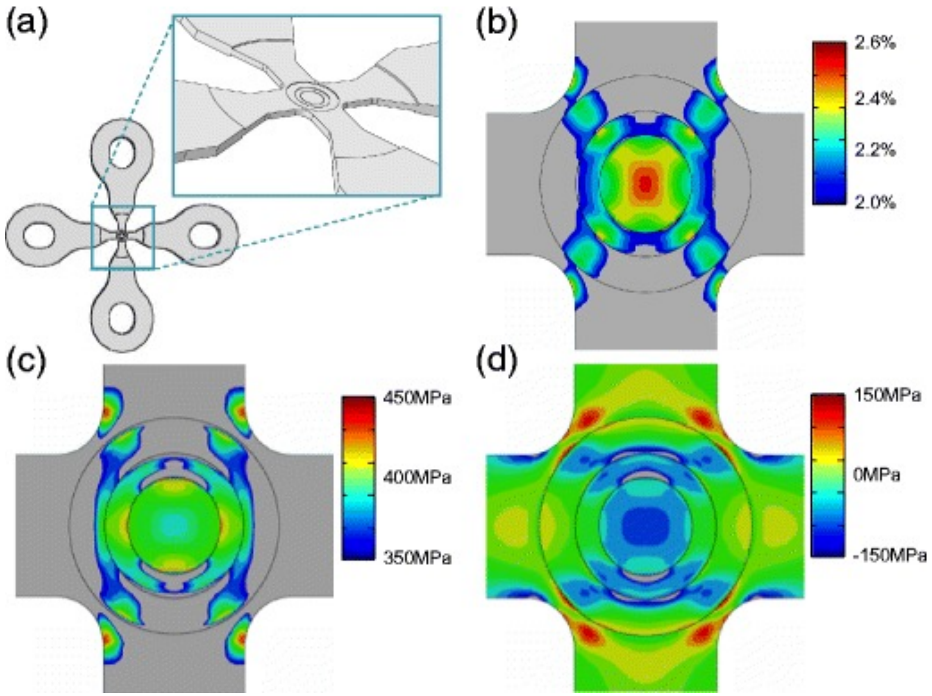


Biaxial tensile testing



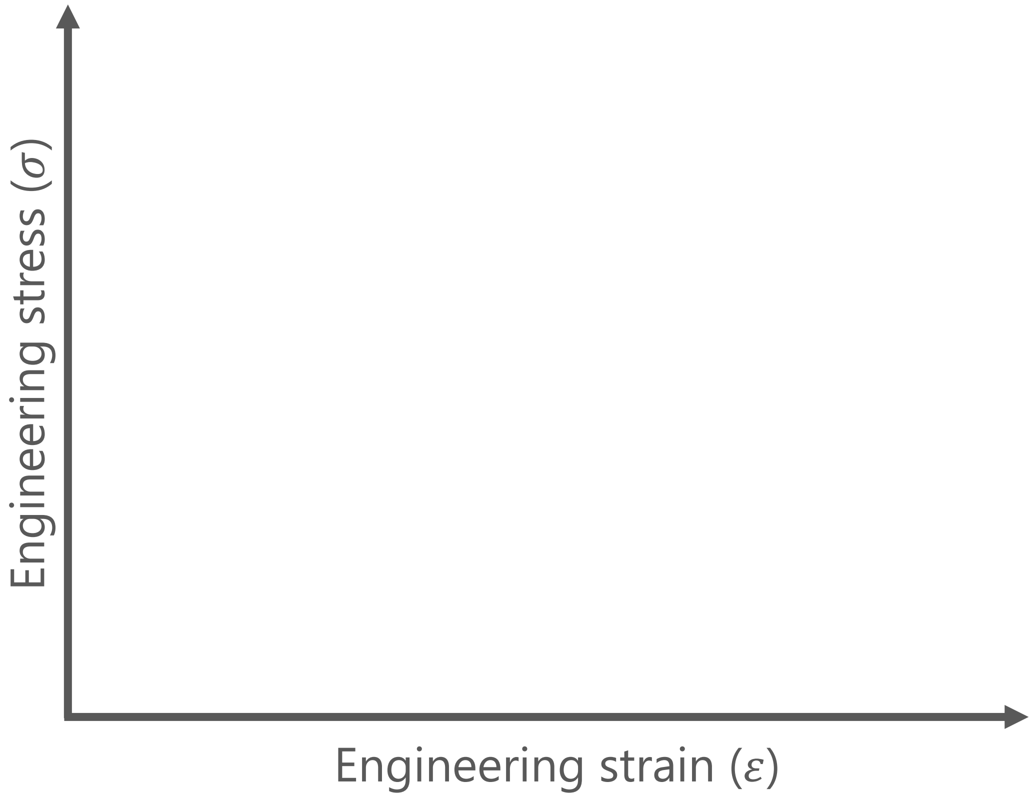




(a) Schematic drawing of the cruciform shaped specimen, (b)-(d) results from a FEM simulation of a uniaxial test up to 2.6% plastic strain showing (b) equivalent plastic strain, (c) stress component along the loading direction and (d) stress component perpendicular to the loading direction

S. Van Petegem et al., *Exp. Mecha.*. (2017)

Use of results



- **Engineering stress-strain curve (ductile material):**
  - The vertical axis represents the engineering stress (σ), defined using the initial cross-sectional area.

$$\sigma = \frac{F_{applied}}{A_{x0}}$$

  - The horizontal axis represents the engineering strain (ε), defined using the initial gauge length.

$$\varepsilon = \frac{\Delta\ell}{\ell_0}$$

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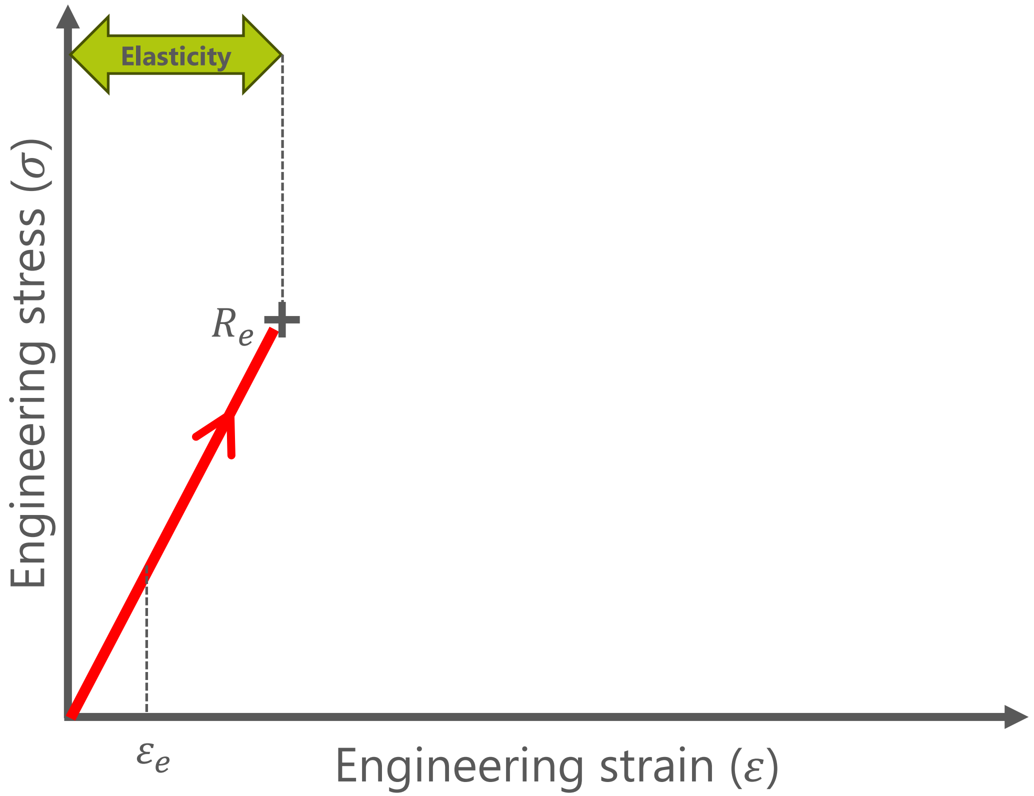

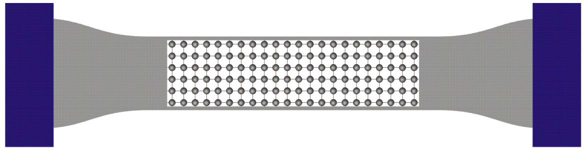

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- **Elastic region ($\sigma < R_e$):**
  - The initial linear part corresponds to the elastic regime.
  - Stress and strain are proportional (Hooke's law):
    $$\sigma = E\varepsilon_e$$
  - Deformation is fully reversible upon unloading.
  - The slope of this region is the Young's modulus (E).
  - The elastic limit is reached at the yield stress, R_e.

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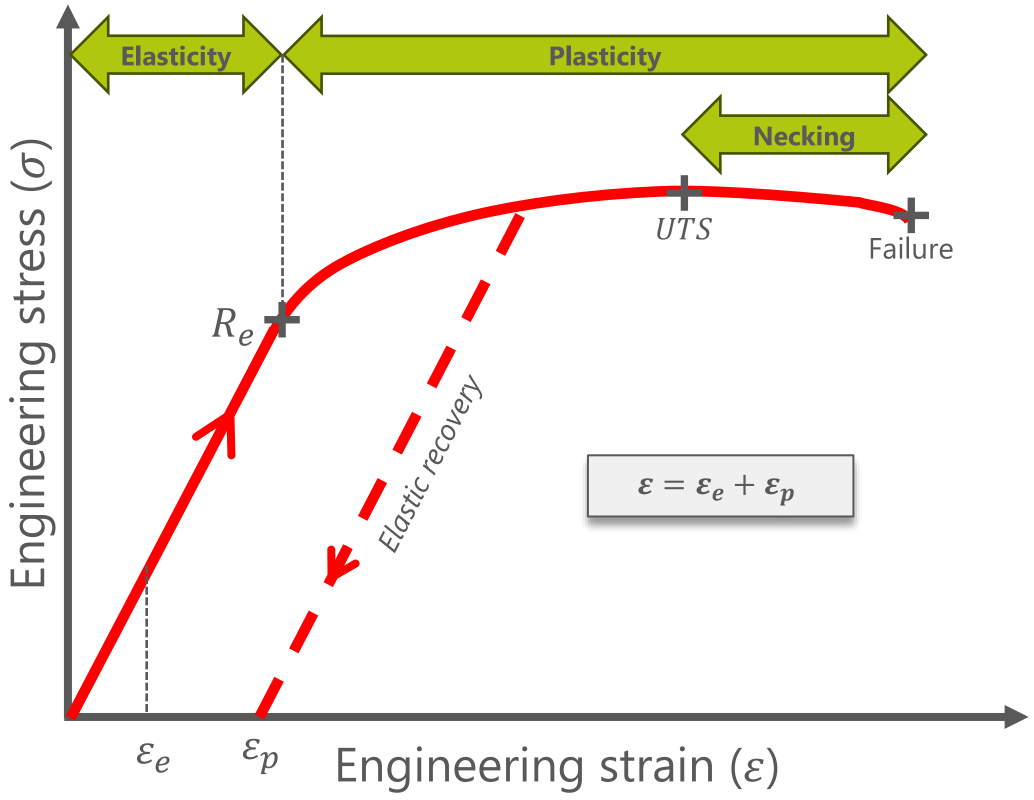

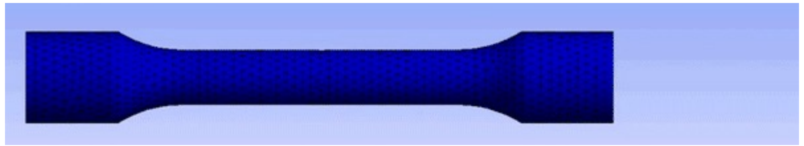

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- **Plastic region ($\sigma > R_e$):**
  - Deformation becomes irreversible.
  - The material undergoes strain hardening, requiring increasing stress to continue deforming.
  - The ultimate tensile strength (UTS) corresponds to the maximum σ and marks the onset of necking.
  - Unloading from this region follows a linear elastic path.
  - After unloading, a permanent plastic strain (ε_p) remains.
  - The recovered part is the elastic strain.

Some vocabulary

- **Elastic behavior**
  - Limit of elasticity
    - ✓ Maximum stress a material can withstand without any permanent deformation.
    - ✓ Conceptual limit; not always easy to measure precisely.

Dieter, G. E., & Bacon, D. J. (2009). *Mechanical Metallurgy* (3rd ed.). London: McGraw-Hill Education

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- **Onset of plasticity**
  - Yield stress
    - ✓ General term for the stress at which plastic deformation begins.
    - ✓ May be theoretical or experimental, but not necessarily standardized.
  - Yield point
    - ✓ A well-defined point where plastic deformation starts suddenly, sometimes followed by a drop in stress.
    - ✓ Observed in specific materials like mild steels, with upper and lower yield points.
  - Yield strength
    - ✓ Standardized engineering value of yield stress.
    - ✓ Typically defined by the 0.2% offset method ($\sigma_{0,2}$), especially for materials without a clear yield point.

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❖ Plastic flow

- Flow stress
  - ✓ Stress required to maintain plastic deformation at a given plastic strain.
  - ✓ Increases with strain hardening; often modeled as $\sigma = K\varepsilon^n$

Dieter, G. E., & Bacon, D. J. (2009). *Mechanical Metallurgy* (3rd ed.). London: McGraw-Hill Education

True stress / true strain

- **True stress**
  - Definition:

$$\sigma_T = \frac{F_{applied}}{A_{xi}}$$

($F_{applied}$: applied force; A_{xi}: instantaneous cross-sectional area)

  - It accounts for the actual load-bearing area as the specimen deforms.
  - Engineering stress uses the initial area
    ⇒ **Underestimates stress at large deformation**
  - Engineering stress drop after necking is a geometric artifact.
  - Required for:
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    - strain hardening laws
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True strain

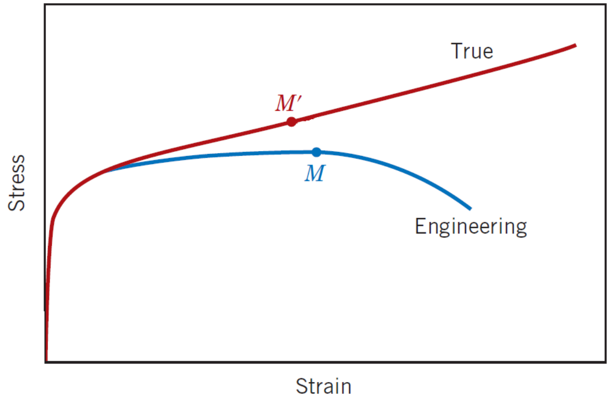
- Definition:

$$\varepsilon_T = \int_{l_0}^{l_i} \frac{dl}{l} = \ln\left(\frac{l_i}{l_0}\right)$$

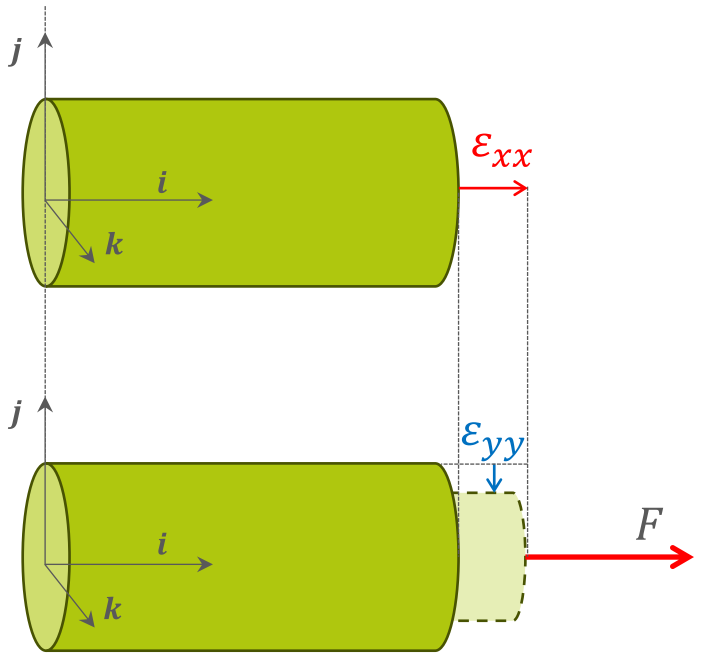
(l_i: instantaneous length; l_0: initial length)

- It is the accumulated sum of incremental deformations.
- Valid for large deformations
- For small deformations: $\varepsilon_T \approx \varepsilon_{eng}$
- Additive for successive deformation steps
- Naturally consistent with material flow and plasticity models
- Relation (before necking)

$$\varepsilon_T = \ln(1 + \varepsilon_{eng})$$



Transversal strain

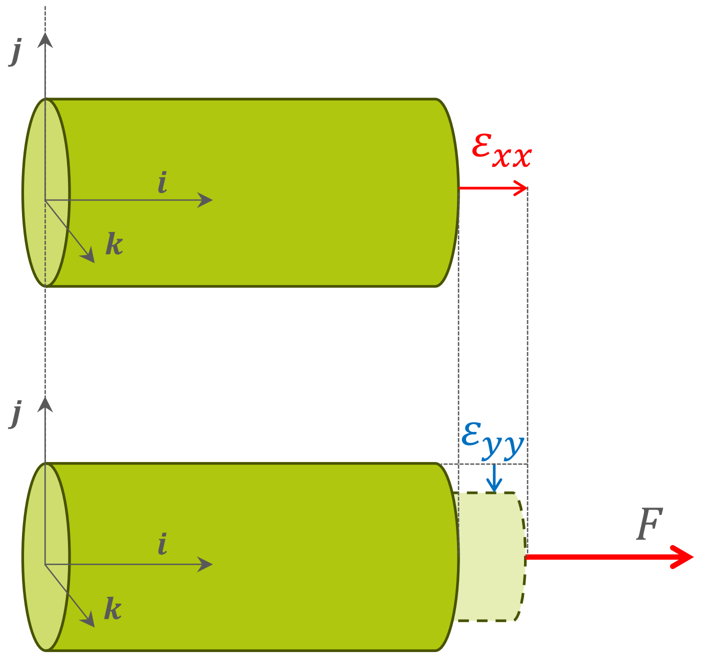


- **Longitudinal strain: ε_{xx}**
- **Transversal strain: ε_{yy} (or ε_{zz})**

$$\varepsilon_{yy} = -\nu\varepsilon_{xx}$$
$$\varepsilon_{zz} = -\nu\varepsilon_{xx}$$

(Poisson's coefficient: $-1 < \nu < 0.5$)

Transversal strain



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$$\varepsilon_{zz} = -\nu\varepsilon_{xx}$$

(Poisson's coefficient: $-1 < \nu < 0.5$)

A linear isotropic elastic/plastic material is characterized by:

- Young's modulus (E)
- Poisson's ratio (ν)
- Yield stress (R_e)

Compression testing

- **Precautions:**
  - If the beam is too long

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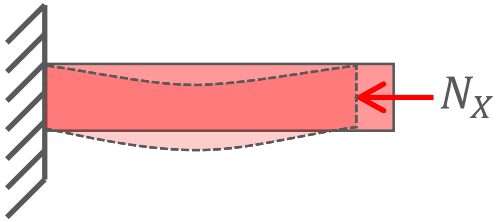

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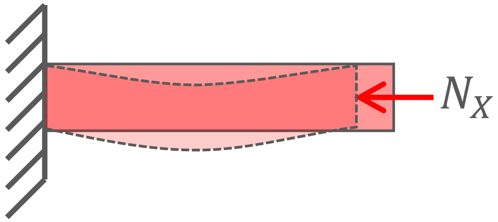

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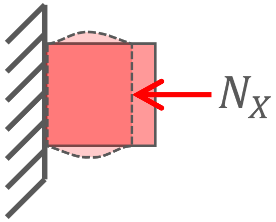

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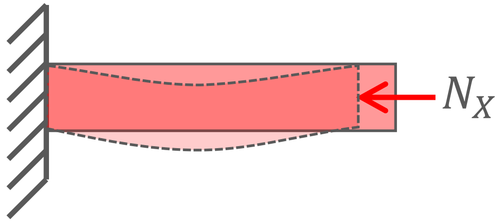

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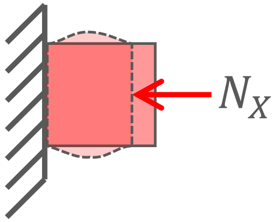
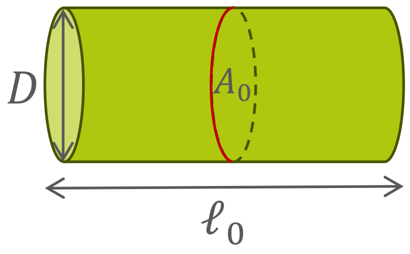


❖ Practical design criteria:

$$3 \lesssim \frac{{\ell_0}^2}{A_{x0}} \lesssim 5$$

$$\frac{\ell_0}{D_0} \sim 1.5 - 2$$



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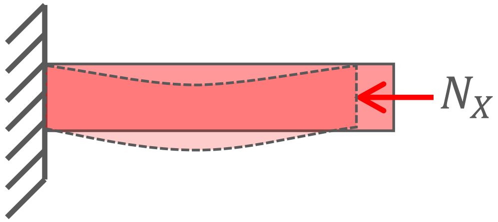

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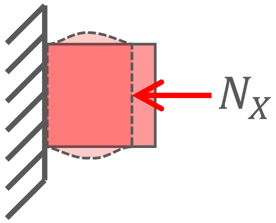

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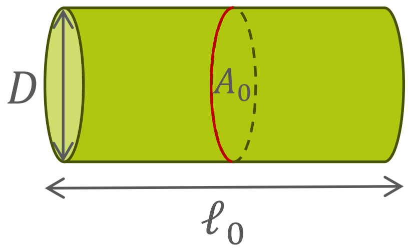
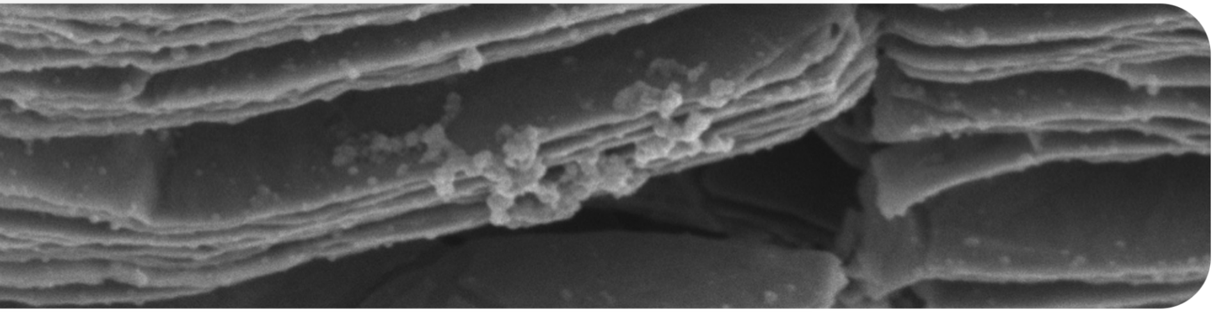


⇘ Brittle materials are much stronger in compression than in tension because compressive stresses tend to close microcracks rather than open them.



Extensions and special cases

Ductile vs. brittle

❖ **Ductile materials:**

- Large plastic deformation before fracture
- Significant elongation and necking

  ⇒ Fracture occurs after extensive dislocation motion

Ductile vs. brittle

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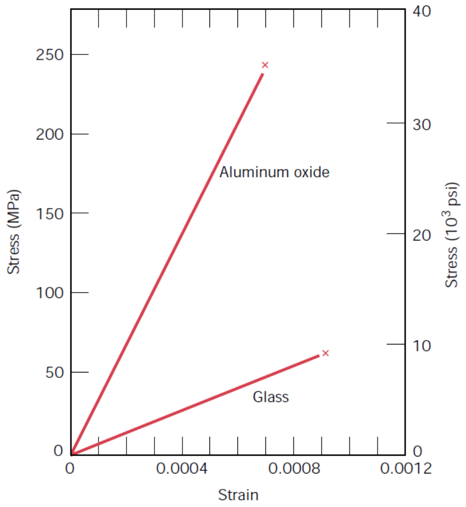

❖ Brittle materials:

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- Sudden fracture once the elastic limit is reached

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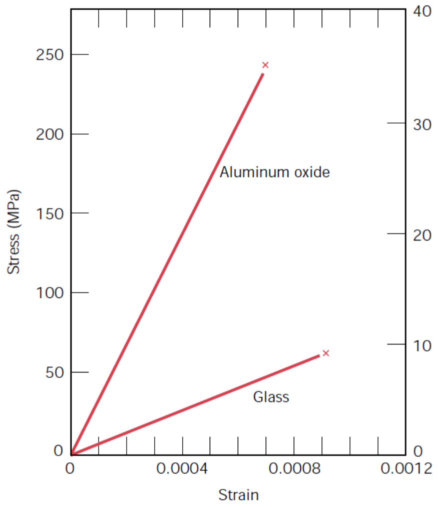
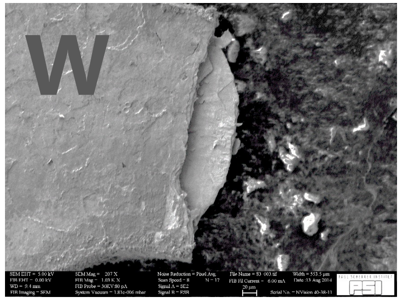
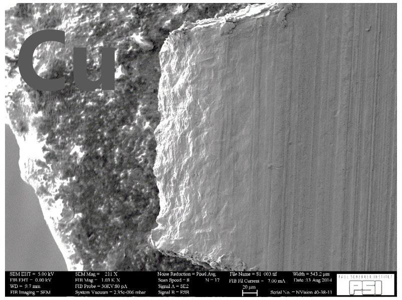
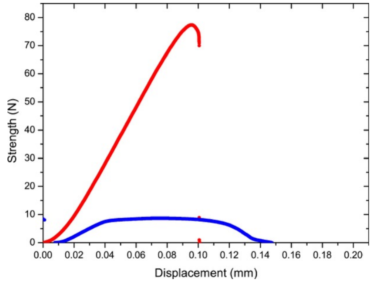
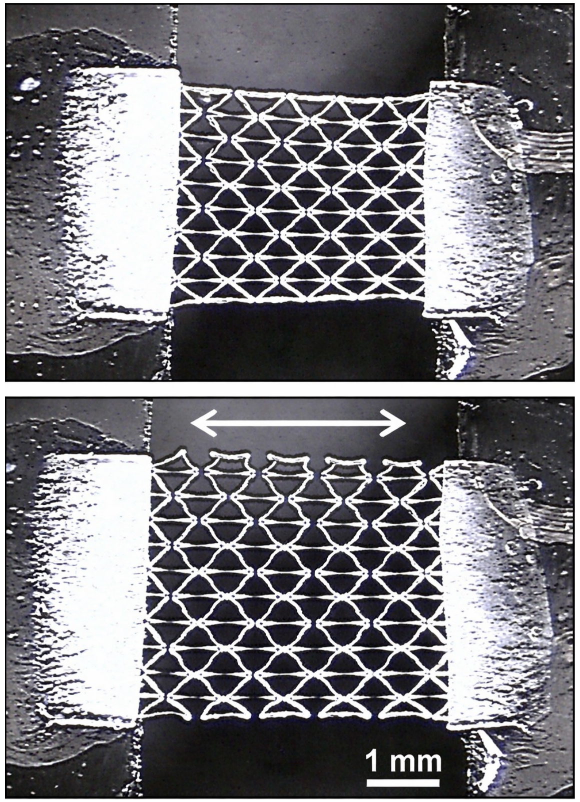








Auxetic behavior

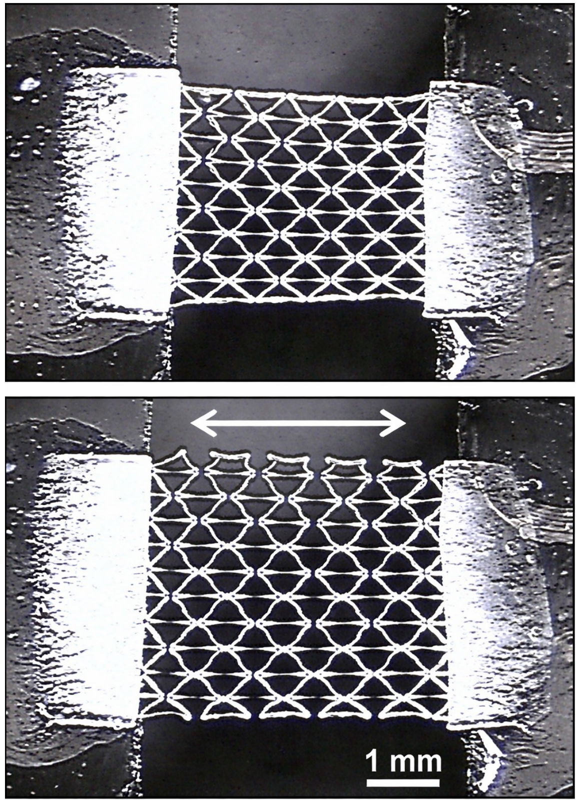


❖ **Auxetic materials are characterized by a negative Poisson's ratio ($\nu < 0$).**

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} < 0$$

C. Sergi, *Polymers MDPI.* (2021)

Auxetic behavior



- **Auxetic materials are characterized by a negative Poisson's ratio ($\nu < 0$).**

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- **Under compression, they contract laterally**

$\varepsilon_{xx} > 0 \Rightarrow \varepsilon_{yy} > 0$

$\varepsilon_{xx} < 0 \Rightarrow \varepsilon_{yy} < 0$

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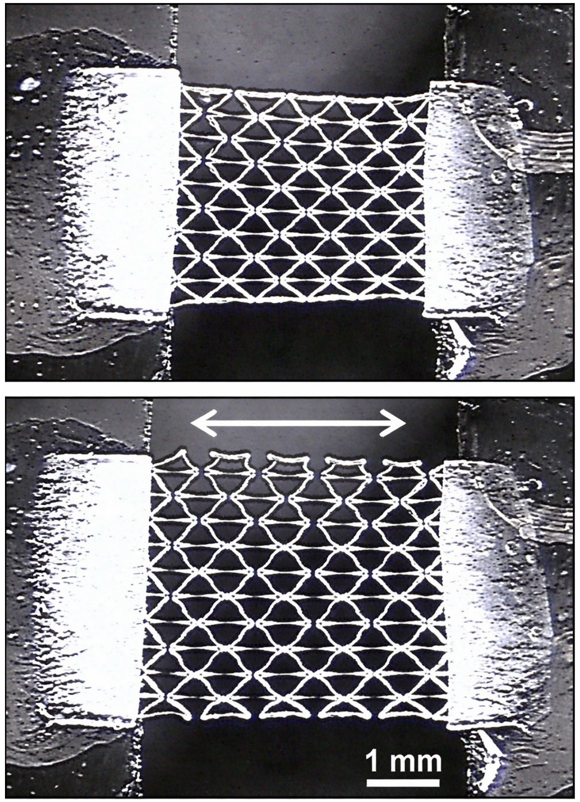
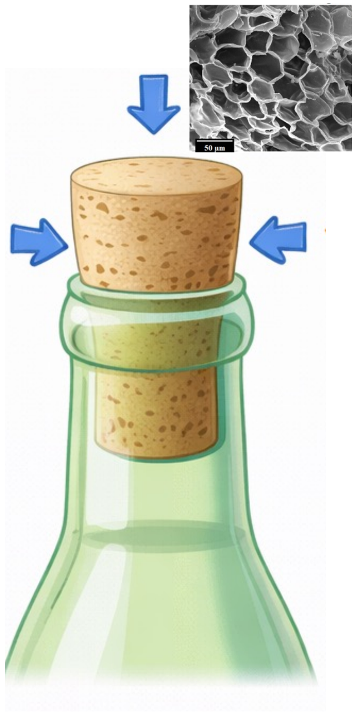
Auxetic behavior





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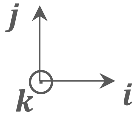
- **Example:**
  - Cork stopper ($\nu \approx -0.1 - 0.1$)

C. Sergi, *Polymers MDPI.* (2021)

Summary

Loadings	Deformation	Stress	Governing equations	Coefficients	Elasticity limit
Tension	$\varepsilon_{xx} > 0$	$\sigma_{xx} = \dfrac{N}{A_x} > 0$	$\sigma_{xx} = E\varepsilon_{xx}$ $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$	E, ν	R_e
Compression	$\varepsilon_{xx} < 0$	$\sigma_{xx} = \dfrac{N}{A_x} < 0$	$\sigma = E\varepsilon_{xx}$ $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$	E, ν	R_e



Thanks for your listening!

If you need further information:

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